Veldkamp Polygons and Tits Polygons

Bernhard Mühlherr (JLU Gießen)

October 8, 2021

This is joint work with Richard Weiss (Tufts University).

The projective plane over $M_k(F)$

Let F be a field and $X := F^{3k}$ for some $1 \leq k \in \mathbb{N}$.
Put

$$\mathcal{P} := \{P \leq_F X \mid \dim_F P = k\} \text{ set of points,}$$

$$\mathcal{L} := \{L \leq_F X \mid \dim_F L = 2k\} \text{ set of lines.}$$

The projective plane over $M_k(F)$

Let F be a field and $X := F^{3k}$ for some $1 \leq k \in \mathbb{N}$.
Put

$$\mathcal{P} := \{P \leq_F X \mid \dim_F P = k\} \text{ set of points,}$$

$$\mathcal{L} := \{L \leq_F X \mid \dim_F L = 2k\} \text{ set of lines.}$$

For any two points $P \neq Q$ there is a line L containing both and the line L is unique if $L = P + Q$.

The projective plane over $M_k(F)$

Let F be a field and $X := F^{3k}$ for some $1 \leq k \in \mathbb{N}$.
Put

$$\mathcal{P} := \{P \leq_F X \mid \dim_F P = k\} \text{ set of points,}$$

$$\mathcal{L} := \{L \leq_F X \mid \dim_F L = 2k\} \text{ set of lines.}$$

For any two points $P \neq Q$ there is a line L containing both and the line L is unique if $L = P + Q$.

For any two lines $L \neq M$ there is a point P contained in both and the point P is unique if $P = L \cap M$.

The projective plane over $M_k(F)$

Let F be a field and $X := F^{3k}$ for some $1 \leq k \in \mathbb{N}$.
Put

$$\mathcal{P} := \{P \leq_F X \mid \dim_F P = k\} \text{ set of points,}$$

$$\mathcal{L} := \{L \leq_F X \mid \dim_F L = 2k\} \text{ set of lines.}$$

For any two points $P \neq Q$ there is a line L containing both and the line L is unique if $L = P + Q$.

For any two lines $L \neq M$ there is a point P contained in both and the point P is unique if $P = L \cap M$.

For $k = 1$ we obtain the projective plane $\mathbf{P}_2(F)$ over F.

The projective plane over $M_k(F)$

Let F be a field and $X := F^{3k}$ for some $1 \leq k \in \mathbb{N}$.
Put

$$\mathcal{P} := \{P \leq_F X \mid \dim_F P = k\} \text{ set of points,}$$

$$\mathcal{L} := \{L \leq_F X \mid \dim_F L = 2k\} \text{ set of lines.}$$

For any two points $P \neq Q$ there is a line L containing both and the line L is unique if $L = P + Q$.

For any two lines $L \neq M$ there is a point P contained in both and the point P is unique if $P = L \cap M$.

For $k = 1$ we obtain the projective plane $\mathbf{P}_2(F)$ over F. For arbitrary k we obtain a point-line geometry that can be interpreted *as the projective plane over the ring* $M_k(F)$.

Projective planes over unital associative rings

Goal: We want to define the projective plane $\mathbf{P}_2(R)$ for an arbitrary unital associative ring R.

Projective planes over unital associative rings

Goal: We want to define the projective plane $\mathbf{P}_2(R)$ for an arbitrary unital associative ring R.

Idea: Let $X = R^3$ be the free module of rank 3 over R and define the points (lines) as the free submodules of rank 1 (2).

Projective planes over unital associative rings

Goal: We want to define the projective plane $\mathbf{P}_2(R)$ for an arbitrary unital associative ring R.

Idea: Let $X = R^3$ be the free module of rank 3 over R and define the points (lines) as the free submodules of rank 1 (2).

For arbitrary rings R this can yield to rather wild point-line geometries that one does not want to call *projective planes*. But for certain classes of rings (local, semi-simple) one obtains meaningful structures.

Projective planes over unital associative rings

Goal: We want to define the projective plane $\mathbf{P}_2(R)$ for an arbitrary unital associative ring R.

Idea: Let $X = R^3$ be the free module of rank 3 over R and define the points (lines) as the free submodules of rank 1 (2).

For arbitrary rings R this can yield to rather wild point-line geometries that one does not want to call *projective planes*. But for certain classes of rings (local, semi-simple) one obtains meaningful structures.

Projective planes over rings were first considered by Segre (1911, for the dual numbers $R = \mathbb{R}[\epsilon]$) and later by various other people (e.g. Hjelmslev, Klingenberg) during the 20th century.

Veldkamp planes

In 1981 Veldkamp observed that the class of rings for which it is possible to develop a meaningful projective geometry is precisely the class of rings of stable rank 2 (a notion from K-theory).

Veldkamp planes

In 1981 Veldkamp observed that the class of rings for which it is possible to develop a meaningful projective geometry is precisely the class of rings of stable rank 2 (a notion from K-theory).

He gave purely combinatorial axioms for point-line geometries that are now known as *Veldkamp planes*. The standard examples are the projective planes $P_2(R)$ where R is a unital, associative ring of stable rank 2.

Veldkamp planes

In 1981 Veldkamp observed that the class of rings for which it is possible to develop a meaningful projective geometry is precisely the class of rings of stable rank 2 (a notion from K-theory).

He gave purely combinatorial axioms for point-line geometries that are now known as *Veldkamp planes*. The standard examples are the projective planes $P_2(R)$ where R is a unital, associative ring of stable rank 2.

Theorem (Veldkamp, 1981): A Veldkamp plane ist Desarguesian if and only if it is the projective plane over a ring of stable rank 2.

Rings of stable rank 2

Let R be a unital associative ring, $R^\times$ the group of units in R and

$$U_+ := \{\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \mid \lambda \in R\} \leq Gl_2(R),$$

$$U_- := \{\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \mid \lambda \in R\} \leq GL_2(R),$$

$$H := \{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in R^\times\} \leq GL_2(R).$$

Rings of stable rank 2

Let R be a unital associative ring, $R^\times$ the group of units in R and

$$U_+ := \{\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \mid \lambda \in R\} \leq Gl_2(R),$$

$$U_- := \{\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \mid \lambda \in R\} \leq GL_2(R),$$

$$H := \{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in R^\times\} \leq GL_2(R).$$

Definition: The ring R is *of stable rank 2* (or just *stable*) if and only if

$$SL_2(R) := \langle U_+, U_- \rangle = U_- U_+ U_- H.$$

Rings of stable rank 2

Let R be a unital associative ring, $R^{\times}$ the group of units in R and

$$U_+ := \{\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \mid \lambda \in R\} \leq Gl_2(R),$$

$$U_- := \{\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \mid \lambda \in R\} \leq GL_2(R),$$

$$H := \{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in R^{\times}\} \leq GL_2(R).$$

Definition: The ring R is *of stable rank 2* (or just *stable*) if and only if

$$SL_2(R) := \langle U_+, U_- \rangle = U_- U_+ U_- H.$$

Observation: Let
$M := \{m \in U_- U_+ U_- \mid U_+^m = U_- \text{ and } U_-^m = U_+\}$, then

$$U_+^{\sharp} := \{u \in U_+ \mid M \cap U_- u U_- \neq \emptyset\} = \{\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \mid \lambda \in R^{\times}\}.$$

Notation for graphs

A (simplicial) graph is a pair $\Gamma = (V, E)$ where V is a set (of *vertices*) and $E \subseteq \binom{V}{2}$ (of *edges*).

Notation for graphs

A (simplicial) graph is a pair $\Gamma = (V, E)$ where V is a set (of *vertices*) and $E \subseteq \binom{V}{2}$ (of *edges*).

For $v \in V$ the set of its neighbors is denoted by Γ_v.

Notation for graphs

A (simplicial) graph is a pair $\Gamma = (V, E)$ where V is a set (of *vertices*) and $E \subseteq \binom{V}{2}$ (of *edges*).

For $v \in V$ the set of its neighbors is denoted by Γ_v.

A *path of length* k is a sequence such that $\{v_{i-1}, v_i\} \in E$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq k$.

Notation for graphs

A (simplicial) graph is a pair $\Gamma = (V, E)$ where V is a set (of *vertices*) and $E \subseteq \binom{V}{2}$ (of *edges*).

For $v \in V$ the set of its neighbors is denoted by Γ_v.

A *path of length* k is a sequence such that $\{v_{i-1}, v_i\} \in E$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq k$.

A *circuit* is a path $\gamma = (v_0, v_1, \ldots, v_k)$ with $k \geq 3$ and $v_0 = v_k$.

Generalized polygons

Let $2 \leq n \in \mathbb{N}$. A *generalized n-gon* is graph Γ such that:

GP1 Γ is bipartite and connected;

GP2 for $1 \leq k < n$ and each k-path $\alpha = (v_0, v_1, \ldots, v_k)$, the path α is the unique path from v_0 to v_k of length at most k;

GP3 every $(n+1)$-path is contained in a $2n$-circuit.

Generalized polygons

Let $2 \leq n \in \mathbb{N}$. A *generalized n-gon* is graph Γ such that:

GP1 Γ is bipartite and connected;

GP2 for $1 \leq k < n$ and each k-path $\alpha = (v_0, v_1, \ldots, v_k)$, the path α is the unique path from v_0 to v_k of length at most k;

GP3 every $(n+1)$-path is contained in a $2n$-circuit.

Remark: Generalized 3-gons are precisely the incidence graphs of projective planes.

Local opposition relations in $\mathbf{P}_2(M_k(F))$

Let F be a field, $X := F^{3k}$ for some $1 \leq k \in \mathbb{N}$ and

$$\mathcal{P} := \{P \leq_F X \mid \dim_F P = k\}, \mathcal{L} := \{L \leq_F X \mid \dim_F L = 2k\}.$$

Local opposition relations in $\mathbf{P}_2(M_k(F))$

Let F be a field, $X := F^{3k}$ for some $1 \leq k \in \mathbb{N}$ and

$$\mathcal{P} := \{P \leq_F X \mid \dim_F P = k\}, \mathcal{L} := \{L \leq_F X \mid \dim_F L = 2k\}.$$

Set $\Gamma := (V, E)$ where

$$V := \mathcal{P} \cup \mathcal{L}, E := \{\{P, L\} \mid P \in \mathcal{P}, L \in \mathcal{L}, P \subseteq L\}.$$

Local opposition relations in $\mathbf{P}_2(M_k(F))$

Let F be a field, $X := F^{3k}$ for some $1 \leq k \in \mathbb{N}$ and

$$\mathcal{P} := \{P \leq_F X \mid \dim_F P = k\}, \mathcal{L} := \{L \leq_F X \mid \dim_F L = 2k\}.$$

Set $\Gamma := (V, E)$ where

$$V := \mathcal{P} \cup \mathcal{L}, E := \{\{P, L\} \mid P \in \mathcal{P}, L \in \mathcal{L}, P \subseteq L\}.$$

For $P \in \mathcal{P}$ define the binary relation $\equiv_P$ on Γ_P

$$L \equiv_P M :\Leftrightarrow L \cap M = P.$$

Local opposition relations in $\mathbf{P}_2(M_k(F))$

Let F be a field, $X := F^{3k}$ for some $1 \leq k \in \mathbb{N}$ and

$$\mathcal{P} := \{P \leq_F X \mid \dim_F P = k\}, \mathcal{L} := \{L \leq_F X \mid \dim_F L = 2k\}.$$

Set $\Gamma := (V, E)$ where

$$V := \mathcal{P} \cup \mathcal{L}, E := \{\{P, L\} \mid P \in \mathcal{P}, L \in \mathcal{L}, P \subseteq L\}.$$

For $P \in \mathcal{P}$ define the binary relation $\equiv_P$ on Γ_P

$$L \equiv_P M :\Leftrightarrow L \cap M = P.$$

For $L \in \mathcal{L}$ define the binary relation $\equiv_L$ on Γ_L

$$P \equiv_L Q :\Leftrightarrow P + Q = L.$$

Local opposition relations in $\mathbf{P}_2(M_k(F))$

Let F be a field, $X := F^{3k}$ for some $1 \leq k \in \mathbb{N}$ and

$$\mathcal{P} := \{P \leq_F X \mid \dim_F P = k\}, \mathcal{L} := \{L \leq_F X \mid \dim_F L = 2k\}.$$

Set $\Gamma := (V, E)$ where

$$V := \mathcal{P} \cup \mathcal{L}, E := \{\{P, L\} \mid P \in \mathcal{P}, L \in \mathcal{L}, P \subseteq L\}.$$

For $P \in \mathcal{P}$ define the binary relation $\equiv_P$ on Γ_P

$$L \equiv_P M :\Leftrightarrow L \cap M = P.$$

For $L \in \mathcal{L}$ define the binary relation $\equiv_L$ on Γ_L

$$P \equiv_L Q :\Leftrightarrow P + Q = L.$$

For $v \in V$ the relation $\equiv_v$ is called *the local opposition relation at* v. The triple $(V, E, (\equiv_v)_{v \in V})$ is a *Veldkamp graph*.

Opposition relations

A binary relation $\equiv$ on a set X is called an *opposition relation* if it is symmetric and anti-reflexive.

Opposition relations

A binary relation $\equiv$ on a set X is called an *opposition relation* if it is symmetric and anti-reflexive.

Examples: $\mathbf{B} = (\mathcal{C}, \delta)$ building of spherical type (W, S) and $\rho \in W$ the longest element in (W, S).

$c \equiv d :\Leftrightarrow \delta(c, d) = \rho$ is an opposition relation on $\mathcal{C}$.

Opposition relations

A binary relation $\equiv$ on a set X is called an *opposition relation* if it is symmetric and anti-reflexive.

Examples: $\mathbf{B} = (\mathcal{C}, \delta)$ building of spherical type (W, S) and $\rho \in W$ the longest element in (W, S).

$c \equiv d :\Leftrightarrow \delta(c, d) = \rho$ is an opposition relation on $\mathcal{C}$.

For $J \subseteq S$ such that $\rho J \rho = J$ one has an opposition relation on the set of all J-residues of $\mathbf{B}$ in a similar fashion.

Opposition relations

A binary relation $\equiv$ on a set X is called an *opposition relation* if it is symmetric and anti-reflexive.

Examples: $\mathbf{B} = (\mathcal{C}, \delta)$ building of spherical type (W, S) and $\rho \in W$ the longest element in (W, S).

$c \equiv d :\Leftrightarrow \delta(c, d) = \rho$ is an opposition relation on $\mathcal{C}$.

For $J \subseteq S$ such that $\rho J \rho = J$ one has an opposition relation on the set of all J-residues of $\mathbf{B}$ in a similar fashion.

As each J-residue R of $\mathbf{B}$ is a building of type $(\langle J \rangle, J)$, the previous yields a (local) opposition relations on the set of chambers of R (set of K-residues contained in R with suitably chosen K).

Opposition relations

Let $\equiv$ be an opposition relation on the set X.

The opposition relation is called *trivial* if $x \equiv y \Leftrightarrow x \neq y$ for all $x, y \in X$.

Opposition relations

Let $\equiv$ be an opposition relation on the set X.

The opposition relation is called *trivial* if $x \equiv y \Leftrightarrow x \neq y$ for all $x, y \in X$.

The opposition relation is called *k-plump* for some $k \in \mathbb{N}$ if for any $x_1, x_2, \ldots, x_k \in X$ there exists $x \in X$ such that $x \equiv x_i$ for $1 \leq i \leq k$.

$$\text{stable} \;:=\; \text{2-plump}$$

Opposition relations

Let $\equiv$ be an opposition relation on the set X.

The opposition relation is called *trivial* if $x \equiv y \Leftrightarrow x \neq y$ for all $x, y \in X$.

The opposition relation is called *k-plump* for some $k \in \mathbb{N}$ if for any $x_1, x_2, \ldots, x_k \in X$ there exists $x \in X$ such that $x \equiv x_i$ for $1 \leq i \leq k$.

$$\text{stable} \; := \text{2-plump}$$

Remark: If **B** is a spherical building defined over a field F, then the local opposition relations from the previous slide are $|F|$-plump.

Veldkamp polygons

A *Veldkamp graph* is a triple $(V, E, (\equiv_v)_{v \in V})$ such that $\Gamma = (V, E)$ is a graph and $\equiv_v$ is a stable opposition relation on Γ_v for each $v \in V$.

Veldkamp polygons

A *Veldkamp graph* is a triple $(V, E, (\equiv_v)_{v \in V})$ such that $\Gamma = (V, E)$ is a graph and $\equiv_v$ is a stable opposition relation on Γ_v for each $v \in V$.

A k-path $\gamma = (v_0, v_1, \ldots, v_k)$ in a Veldkamp graph Γ is called *straight*, if $v_{i-1} \equiv_{v_i} v_{i+1}$ for all $1 \leq i \leq k-1$.

Veldkamp polygons

A *Veldkamp graph* is a triple $(V, E, (\equiv_v)_{v \in V})$ such that $\Gamma = (V, E)$ is a graph and $\equiv_v$ is a stable opposition relation on Γ_v for each $v \in V$.

A k-path $\gamma = (v_0, v_1, \ldots, v_k)$ in a Veldkamp graph Γ is called *straight*, if $v_{i-1} \equiv_{v_i} v_{i+1}$ for all $1 \leq i \leq k-1$.

Definition: Let $2 \leq n \in \mathbb{N}$. A *Veldkamp n-gon* is a Veldkamp graph Γ such that:

- VP1 Γ is bipartite and connected;
- VP2 for $1 \leq k < n$ and each **straight** k-path $\alpha = (v_0, v_1, \ldots, v_k)$, the path α is the unique **straight** path from v_0 to v_k of length at most k;
- VP3 every **straight** $(n+1)$-path is contained in a **straight** $2n$-circuit.

Examples and remarks

The Veldkamp triangles (i.e 3-gons) are precisely the incidence graphs of Veldkamp planes.

Examples and remarks

The Veldkamp triangles (i.e 3-gons) are precisely the incidence graphs of Veldkamp planes.

A Veldkamp n-gon is a generalized n-gon if and only if all its local opposition relations are trivial.

Examples and remarks

The Veldkamp triangles (i.e 3-gons) are precisely the incidence graphs of Veldkamp planes.

A Veldkamp n-gon is a generalized n-gon if and only if all its local opposition relations are trivial.

Let $S = (\mathcal{P}, \mathcal{L})$ be a non-degenerate polar space of rank at least 2. Then the incidence graph (on points and lines) is a Veldkamp quadrangle.

Examples and remarks

The Veldkamp triangles (i.e 3-gons) are precisely the incidence graphs of Veldkamp planes.

A Veldkamp *n*-gon is a generalized *n*-gon if and only if all its local opposition relations are trivial.

Let $S = (\mathcal{P}, \mathcal{L})$ be a non-degenerate polar space of rank at least 2. Then the incidence graph (on points and lines) is a Veldkamp quadrangle.

Further interesting examples arise from **foldings of spherical buildings**.

Examples and remarks

The Veldkamp triangles (i.e 3-gons) are precisely the incidence graphs of Veldkamp planes.

A Veldkamp *n*-gon is a generalized *n*-gon if and only if all its local opposition relations are trivial.

Let $S = (\mathcal{P}, \mathcal{L})$ be a non-degenerate polar space of rank at least 2. Then the incidence graph (on points and lines) is a Veldkamp quadrangle.

Further interesting examples arise from **foldings of spherical buildings**.

Examples and remarks

The Veldkamp triangles (i.e 3-gons) are precisely the incidence graphs of Veldkamp planes.

A Veldkamp *n*-gon is a generalized *n*-gon if and only if all its local opposition relations are trivial.

Let $S = (\mathcal{P}, \mathcal{L})$ be a non-degenerate polar space of rank at least 2. Then the incidence graph (on points and lines) is a Veldkamp quadrangle.

Further interesting examples arise from **foldings of spherical buildings**.

These are dihedral subgroups of finite Coxeter groups, that are *isometrically embedded*.

J-compatible subsets of Coxeter groups

Let (W, S) be a Coxeter system and for $w \in W$ set

$$S^+(w) := \{s \in S \mid \ell(ws) = \ell(w) + 1\}, S^-(w) := S \setminus S^+(w).$$

J-compatible subsets of Coxeter groups

Let (W, S) be a Coxeter system and for $w \in W$ set

$$S^+(w) := \{s \in S \mid \ell(ws) = \ell(w) + 1\}, S^-(w) := S \setminus S^+(w).$$

Definition: Let $J \subseteq S$.

An element $w \in W$ is J-compatible if $J \subseteq S^+(w)$ or $J \subseteq S^-(w)$.

J-compatible subsets of Coxeter groups

Let (W, S) be a Coxeter system and for $w \in W$ set

$$S^+(w) := \{s \in S \mid \ell(ws) = \ell(w) + 1\}, S^-(w) := S \setminus S^+(w).$$

Definition: Let $J \subseteq S$.

An element $w \in W$ is J-compatible if $J \subseteq S^+(w)$ or $J \subseteq S^-(w)$.

A subset X of W is J-compatible if each $x \in X$ is J-compatible.

Foldings of spherical Coxeter systems

Let (W, S) be a spherical Coxeter system and for each $J \subseteq S$ let ρ_J denote the longest element of $(\langle J \rangle, J)$.

Foldings of spherical Coxeter systems

Let (W, S) be a spherical Coxeter system and for each $J \subseteq S$ let ρ_J denote the longest element of $(\langle J \rangle, J)$.

Let $J, K \subseteq S$ be such that $J \neq \emptyset \neq K$ and $J \cap K = \emptyset$. Put

$$A := S \setminus (J \cup K), r_J := \rho_{J \cup A} \text{ and } r_K := \rho_{K \cup A}.$$

Then the pair (J, K) is called a folding of (W, S) if the following hold:

Fo1 $[r_J, \rho_A] = 1 = [r_K, \rho_A]$;

Foldings of spherical Coxeter systems

Let (W, S) be a spherical Coxeter system and for each $J \subseteq S$ let ρ_J denote the longest element of $(\langle J \rangle, J)$.

Let $J, K \subseteq S$ be such that $J \neq \emptyset \neq K$ and $J \cap K = \emptyset$. Put

$$A := S \setminus (J \cup K), r_J := \rho_{J \cup A} \text{ and } r_K := \rho_{K \cup A}.$$

Then the pair (J, K) is called a folding of (W, S) if the following hold:

Fo1 $[r_J, \rho_A] = 1 = [r_K, \rho_A]$;

Fo2 $\langle r_J, r_K \rangle$ is J-compatible and K-compatible.

The *gonality of a folding* (J, K) is the order of $r_J r_K$.

Examples

$(\{k\}, \{2k\})$ is a folding of gonality 3 of A_{3k-1}.

Examples

$(\{k\}, \{2k\})$ is a folding of gonality 3 of A_{3k-1}.

For $n \geq 3$ the pair $(\{1, n\}, \{2, n-1\})$ is a folding of gonality 4 of A_n.

Examples

$(\{k\}, \{2k\})$ is a folding of gonality 3 of A_{3k-1}.

For $n \geq 3$ the pair $(\{1, n\}, \{2, n-1\})$ is a folding of gonality 4 of A_n.

The pair $(\{1\}, \{2\})$ is a folding of gonality 4 of C_n for $n \geq 2$ and of D_n for $n \geq 4$.

Examples

$(\{k\}, \{2k\})$ is a folding of gonality 3 of A_{3k-1}.

For $n \geq 3$ the pair $(\{1, n\}, \{2, n-1\})$ is a folding of gonality 4 of A_n.

The pair $(\{1\}, \{2\})$ is a folding of gonality 4 of C_n for $n \geq 2$ and of D_n for $n \geq 4$.

The pair $(\{2\}, \{1, 3\}$ is a folding of gonality 6 of C_n for $n \geq 3$ and of D_n for $n \geq 5$.

Examples

$(\{k\}, \{2k\})$ is a folding of gonality 3 of A_{3k-1}.

For $n \geq 3$ the pair $(\{1, n\}, \{2, n-1\})$ is a folding of gonality 4 of A_n.

The pair $(\{1\}, \{2\})$ is a folding of gonality 4 of C_n for $n \geq 2$ and of D_n for $n \geq 4$.

The pair $(\{2\}, \{1, 3\}$ is a folding of gonality 6 of C_n for $n \geq 3$ and of D_n for $n \geq 5$.

Tits indices of relative rank 2 provide examples of foldings.

Examples

$(\{k\}, \{2k\})$ is a folding of gonality 3 of A_{3k-1}.

For $n \geq 3$ the pair $(\{1, n\}, \{2, n-1\})$ is a folding of gonality 4 of A_n.

The pair $(\{1\}, \{2\})$ is a folding of gonality 4 of C_n for $n \geq 2$ and of D_n for $n \geq 4$.

The pair $(\{2\}, \{1, 3\}$ is a folding of gonality 6 of C_n for $n \geq 3$ and of D_n for $n \geq 5$.

Tits indices of relative rank 2 provide examples of foldings.

The bipartite partition of an irreducible spherical diagram is a folding of gonality h (Coxeter number).

Veldkamp polygons from foldings of spherical buildings

Let (W, S) be a spherical Coxeter system, (J, K) a folding of (W, S) of gonality n and Δ a building of type (W, S) (viewed as a simplicial complex).

Veldkamp polygons from foldings of spherical buildings

Let (W, S) be a spherical Coxeter system, (J, K) a folding of (W, S) of gonality n and Δ a building of type (W, S) (viewed as a simplicial complex).

Let $\mathcal{S}_J$ (resp. $\mathcal{S}_K$) be the set of simplices of type J (resp. K) and put

$$V := \mathcal{S}_J \cup \mathcal{S}_K, E := \{\{A, B\} \in \binom{V}{2} \mid A \cup B \text{ simplex of type } J \cup K\}.$$

Veldkamp polygons from foldings of spherical buildings

Let (W, S) be a spherical Coxeter system, (J, K) a folding of (W, S) of gonality n and Δ a building of type (W, S) (viewed as a simplicial complex).

Let $\mathcal{S}_J$ (resp. $\mathcal{S}_K$) be the set of simplices of type J (resp. K) and put

$$V := \mathcal{S}_J \cup \mathcal{S}_K, E := \{\{A, B\} \in \binom{V}{2} \mid A \cup B \text{ simplex of type } J \cup K\}.$$

At each vertex $v \in V$ there is a natural oppostition relation $\equiv_v$ of Γ_v inherited from Δ.

Veldkamp polygons from foldings of spherical buildings

Let (W, S) be a spherical Coxeter system, (J, K) a folding of (W, S) of gonality n and Δ a building of type (W, S) (viewed as a simplicial complex).

Let $\mathcal{S}_J$ (resp. $\mathcal{S}_K$) be the set of simplices of type J (resp. K) and put

$$V := \mathcal{S}_J \cup \mathcal{S}_K, E := \{\{A, B\} \in \binom{V}{2} \mid A \cup B \text{ simplex of type } J \cup K\}.$$

At each vertex $v \in V$ there is a natural oppostition relation $\equiv_v$ of Γ_v inherited from Δ.

Observation 1: $\Gamma = (V, E, (\equiv_v)_{v \in V})$ is a Veldkamp n-gon.

Veldkamp polygons from foldings of spherical buildings

Let (W, S) be a spherical Coxeter system, (J, K) a folding of (W, S) of gonality n and Δ a building of type (W, S) (viewed as a simplicial complex).

Let $\mathcal{S}_J$ (resp. $\mathcal{S}_K$) be the set of simplices of type J (resp. K) and put

$$V := \mathcal{S}_J \cup \mathcal{S}_K, E := \{\{A, B\} \in \binom{V}{2} \mid A \cup B \text{ simplex of type } J \cup K\}.$$

At each vertex $v \in V$ there is a natural oppostition relation $\equiv_v$ of Γ_v inherited from Δ.

Observation 1: $\Gamma = (V, E, (\equiv_v)_{v \in V})$ is a Veldkamp n-gon.

Observation 2: If Δ is Moufang, then Γ is Moufang.

Tits polygons

Let $\Gamma = (V, E, (\equiv_v)_{v \in V})$ be a Veldkamp n-gon.

A *root of* Γ is a straight n-path and an *apartment of* Γ is a straight $2n$-circuit.

Tits polygons

Let $\Gamma = (V, E, (\equiv_v)_{v \in V})$ be a Veldkamp n-gon.

A *root of* Γ is a straight n-path and an *apartment of* Γ is a straight $2n$-circuit.

Observation: For each root α the corresponding root group U_α acts freely on the apartments containing α.

Tits polygons

Let $\Gamma = (V, E, (\equiv_v)_{v \in V})$ be a Veldkamp n-gon.

A *root of* Γ is a straight n-path and an *apartment of* Γ is a straight $2n$-circuit.

Observation: For each root α the corresponding root group U_α acts freely on the apartments containing α.

Definition: Γ is called a Tits polygon if U_α is transitive on the apartements containing α for each root α.

Tits triangles

Theorem: A 5-plump Tits triangle is isomorphic to the projective plane $P_2(A)$ for some stable unital alternative ring.

Tits triangles

Theorem: A 5-plump Tits triangle is isomorphic to the projective plane $P_2(A)$ for some stable unital alternative ring.

Remark: This improves and unifies results of Veldkamp on Veldkamp planes and of Faulkner on A_2-graded groups from the 1980s.

Classifying Tits polygons

Plan: Follow the strategy of Tits and Weiss in their classification of Moufang polygons

Classifying Tits polygons

Plan: Follow the strategy of Tits and Weiss in their classification of Moufang polygons

Question: What about $n = 3, 4, 6$ and 8?

Classifying Tits polygons

Plan: Follow the strategy of Tits and Weiss in their classification of Moufang polygons

Question: What about $n = 3, 4, 6$ and 8?

Fact: There are Tits n-gons for all $3 \leq n \in \mathbb{N}$ (via foldings of $A_{n-1}(F)$).

Classifying Tits polygons

Plan: Follow the strategy of Tits and Weiss in their classification of Moufang polygons

Question: What about $n = 3, 4, 6$ and 8?

Fact: There are Tits n-gons for all $3 \leq n \in \mathbb{N}$ (via foldings of $A_{n-1}(F)$).

Goal: Rule them out by a suitable condition.

Sharp Tits polygons

Remark: For each root group U_α we can define the set $U_\alpha^\sharp \subseteq U_\alpha$ of its *invertible elements*.

Sharp Tits polygons

Remark: For each root group U_α we can define the set $U_\alpha^\sharp \subseteq U_\alpha$ of its *invertible elements*.

Definition: Let Γ be a Tits polygon, Σ an apartment of Γ and T the pointwise stabilizer of Σ in $\mathrm{Aut}(\Gamma)$.

Sharp Tits polygons

Remark: For each root group U_α we can define the set $U_\alpha^\sharp \subseteq U_\alpha$ of its *invertible elements*.

Definition: Let Γ be a Tits polygon, Σ an apartment of Γ and T the pointwise stabilizer of Σ in $\mathrm{Aut}(\Gamma)$.
Then Γ is called *sharp* if the following condition is statisfied for each root α contained in Σ:

Sh If $1 \neq W$ is an H-invariant, normal subgroup of U_α, then

$$U_\alpha^\sharp \cap W \neq \emptyset.$$

Sharp Tits polygons

Remark: For each root group U_α we can define the set $U_\alpha^\sharp \subseteq U_\alpha$ of its *invertible elements*.

Definition: Let Γ be a Tits polygon, Σ an apartment of Γ and T the pointwise stabilizer of Σ in $\mathrm{Aut}(\Gamma)$.
Then Γ is called *sharp* if the following condition is statisfied for each root α contained in Σ:

Sh If $1 \neq W$ is an H-invariant, normal subgroup of U_α, then

$$U_\alpha^\sharp \cap W \neq \emptyset.$$

Remark: For an stable unital associative ring R we have

$$P_2(R) \text{ is sharp } \Leftrightarrow R \text{ is simple }.$$

On the classification of sharp Tits polygons

For a sharp Tits n-gon Γ the following holds:

1. Γ sharp $\Rightarrow n \in \{3, 4, 6, 8\}$;

On the classification of sharp Tits polygons

For a sharp Tits n-gon Γ the following holds:

1. Γ sharp $\Rightarrow n \in \{3, 4, 6, 8\}$;
2. **n=3:** $\Gamma = \mathbf{P}_2(A)$ for a simple, stable unital alternative ring A;

On the classification of sharp Tits polygons

For a sharp Tits n-gon Γ the following holds:

1. Γ sharp $\Rightarrow n \in \{3, 4, 6, 8\}$;
2. **n=3:** $\Gamma = \mathbf{P}_2(A)$ for a simple, stable unital alternative ring A;
3. **n=6:** Non-degenerate cubic norm forms;

On the classification of sharp Tits polygons

For a sharp Tits n-gon Γ the following holds:

1. Γ sharp $\Rightarrow n \in \{3, 4, 6, 8\}$;
2. **n=3:** $\Gamma = \mathbf{P}_2(A)$ for a simple, stable unital alternative ring A;
3. **n=6:** Non-degenerate cubic norm forms;
4. **n=8:** Tits octagons are precisely the Moufang octagons;

On the classification of sharp Tits polygons

For a sharp Tits n-gon Γ the following holds:

1. Γ sharp $\Rightarrow n \in \{3, 4, 6, 8\}$;
2. **n=3:** $\Gamma = \mathbf{P}_2(A)$ for a simple, stable unital alternative ring A;
3. **n=6:** Non-degenerate cubic norm forms;
4. **n=8:** Tits octagons are precisely the Moufang octagons;
5. **n=4 exceptional:** Quadrangular algebras;

On the classification of sharp Tits polygons

For a sharp Tits n-gon Γ the following holds:

1. Γ sharp $\Rightarrow n \in \{3, 4, 6, 8\}$;
2. **n=3:** $\Gamma = \mathbf{P}_2(A)$ for a simple, stable unital alternative ring A;
3. **n=6:** Non-degenerate cubic norm forms;
4. **n=8:** Tits octagons are precisely the Moufang octagons;
5. **n=4 exceptional:** Quadrangular algebras;
6. **n=4 non-exceptional:** in progress.

On the classification of sharp Tits polygons

For a sharp Tits n-gon Γ the following holds:

1. Γ sharp $\Rightarrow n \in \{3, 4, 6, 8\}$;
2. **n=3:** $\Gamma = \mathbf{P}_2(A)$ for a simple, stable unital alternative ring A;
3. **n=6:** Non-degenerate cubic norm forms;
4. **n=8:** Tits octagons are precisely the Moufang octagons;
5. **n=4 exceptional:** Quadrangular algebras;
6. **n=4 non-exceptional:** in progress.

Remark: With a few obvious exceptions, all examples is provided by a folding of a higher rank building, where the folding is associated with a Tits index of relative rank 2.

Concluding Remarks

A question: Do all Tits n-gons for $n \notin \{3, 4\}$ arise from foldings of spherical buildings?

Concluding Remarks

A question: Do all Tits *n*-gons for $n \notin \{3,4\}$ arise from foldings of spherical buildings?

Our original motivation: Each exceptional Moufang set can be realized as fixed point set of a Galois involution on a Tits quadrangle or a Tits hexagon.

Concluding Remarks

A question: Do all Tits n-gons for $n \notin \{3, 4\}$ arise from foldings of spherical buildings?

Our original motivation: Each exceptional Moufang set can be realized as fixed point set of a Galois involution on a Tits quadrangle or a Tits hexagon.

Root graded groups: Let $3 \leq n \in \mathbb{N}$ and $(W, S) = (D_{2n}, \{s, t\})$

$$\text{Moufang } n\text{-gons} \;\Leftrightarrow\; RGD\text{-systems of type } (W, S),$$

$$\text{Tits } n\text{-gons} \;\Leftrightarrow\; \text{groups with a stable root grading of type } (W, S).$$

Concluding Remarks

A question: Do all Tits n-gons for $n \notin \{3, 4\}$ arise from foldings of spherical buildings?

Our original motivation: Each exceptional Moufang set can be realized as fixed point set of a Galois involution on a Tits quadrangle or a Tits hexagon.

Root graded groups: Let $3 \leq n \in \mathbb{N}$ and $(W, S) = (D_{2n}, \{s, t\})$

$$\text{Moufang } n\text{-gons} \;\Leftrightarrow\; RGD\text{-systems of type } (W, S),$$

$$\text{Tits } n\text{-gons} \;\Leftrightarrow\; \text{groups with a stable root grading of type } (W, S).$$

Veldkamp buildings: The definition of a Veldkamp polygon can be easily modified to define Veldkamp buildings of type (W, S). Presumably one can classify the irreducible spherical buildings of rank at least 3.