Part 1

Sonntag, 19. September 2021 20:30

Random subcomplexes of finite buildings and applications to RACGs

Joint work with M. Zaremsky.

- I Finiteness properties

- Definition: A group G is of type F_n if it acts freely on a contractible cell complex X, s.t. X/G has compact n-skeleton.

- $F_1 \Leftrightarrow$ finite generation, $F_2 \Leftrightarrow$ finite presentability

- Definition: A group G algebraically F_n-fibers if there is some epimorphism $\varphi: G \twoheadrightarrow \mathbb{Z}$, s.t. $\ker(\varphi)$ is of type F_n.

- Question: Which RACG's W_L virtually algebraically F_n-fiber?

- Fact: If T is a tree with at least 3 vertices, then A_T algebraically F_∞-fibers, while W_T does not even virtual F_1-fiber.

Part 2

Dienstag, 28. September 2021 16:21

II The JNW game

- Let L be a finite flag complex and let $\mathcal{P}(L)$ denote the set of induced subcomplexes $X \leq L$.

$\forall \sigma \subseteq L$ simplex : $\sigma \subseteq X \Leftrightarrow \sigma^{(0)} \subseteq X$.

- Def: A subset $\sigma \subseteq L^{(0)}$ is called a state. A state σ is k-legal for some $k \in \mathbb{N}_0$, if σ and $L^{(0)} \setminus \sigma$ induce k-connected subcomplexes of L.

- For every $v \in L^{(0)}$ we define a move as a state $\mu_v \subseteq L^{(0)}$, s.t. $v \in \mu_v$ and $w \notin \mu_v$ if w is adjacent to v. We call $\{\mu_v \mid v \in L^{(0)}\}$ a system of moves.

- Example: Let $\chi(L)$ be the chromatic number of L and let $c : L^{(0)} \to \{1, \dots, \chi(L)\}$ be a coloring. Then $v \mapsto \mu_v := c^{-1}(\chi(v))$ defines a system of moves.

- Observation: The power set $\mathcal{P}(L^{(0)})$ associated with the symmetric difference Δ is isomorphic to $W_L^{ab} = W_L / W_L'$ via $\sigma \mapsto \sum_{v \in \sigma^{(0)}} v$.

- Def: A system of moves $M = \{\mu_v \mid v \in L^{(0)}\}$

is called k-legal, $k \in \mathbb{N}_0$ if there is a coset in $\mathcal{P}(\Delta)/\langle M \rangle$ consisting of k-legal states.

- Theorem (JNW): If L admits an $(m-1)$-legal system of moves, then W_L' algebraically F_m-fibers.

Proof: Let $\sigma \subseteq L^{(0)}$, $M = \{\mu_v \mid v \in L^{(0)}\}$ be s.t. $\sigma + \mu$ is $(m-1)$-legal $\forall \mu \in \langle M \rangle$.

Goal: Define $h: W_L \to \mathbb{Z}$, s.t. $h: W_L' \to \mathbb{Z}$ is a homomorphism.

$h(1) = 0$

$\forall v \in L^{(0)}: h(v) = 1 \Leftrightarrow v \in \sigma,\ h(v) = -1 \Leftrightarrow v \in L^{(0)} \setminus \sigma$

$h(vw) = h(v) + \begin{cases} 1 ; & w \in \sigma \Delta \mu_v \\ -1 ; & w \notin \sigma \Delta \mu_v \end{cases}$

$h(v_1 \cdots v_n v_{n+1}) = h(v_1 \cdots v_n) + \begin{cases} 1 ; & v_{n+1} \in \sigma + \mu_{v_1} + \dots + \mu_{v_n} \\ -1 ; & v_{n+1} \notin \text{"} \end{cases}$

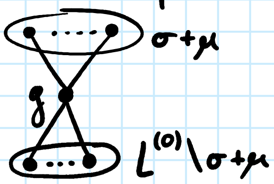
h is well defined: Suppose $\overset{v}{\bullet}\!\!-\!\!\!-\!\!\overset{w}{\bullet} \subseteq L^{(1)}$.

$$h(gwv) - h(gw) > 0$$
$\Leftrightarrow v \in \sigma + \mu + \mu_w$
$\Leftrightarrow v \in \sigma + \mu \qquad (v \notin \mu_w)$
$\Leftrightarrow h(gv) - h(g) > 0$
$\Rightarrow h(gvw) = h(gwv)$

Observation: If $v_1 \cdots v_k \in W_L'$ then each generator occurs an even number of times.
$\Rightarrow \sum_{i=1}^{k} \mu_{v_i} = 0$
$\Rightarrow \forall h \in W_L': h(hg) = h(h) + h(g)$
$\Rightarrow h: W_L' \to \mathbb{Z}$ is an epimorphism.

Extend h to the Davis complex $h: X_L \to \mathbb{R}$.

$\rightsquigarrow W_L' \curvearrowright X_L$ h-equivariant: $h(gx) = h(g) + h(x)$.

Links: 

$\underset{\text{Morse theory}}{\rightsquigarrow}$ $\ker(\chi)$ is of type F_n.

- Italiano, Martelli, Migliorini (2021): There is a hyperbolic group G that contains a subgroup H of type F that is not hyperbolic.

Part 3

Freitag, 1. Oktober 2021 23:29

- Def.: A k-legal state $\sigma \subseteq L^{(0)}$ is sharply k-legal if $\sigma, \sigma^c := L^{(0)} \setminus \sigma$ induce subcomplexes that are neither $(k+1)$-acyclic and have trivial $(k+2)$-homology.

- Lemma: If L admits a sharply $(m-1)$-legal system of moves, then there is some morphism $W_L' \to \mathbb{Z}$ whose kernel is F_m but not FP_{m+1}.

- Def.: For $k \geq -1$ let
  $\mathcal{F}_k(L) := \{X \in \mathcal{P}(L) \mid X \text{ not } k\text{-connected}\}$
  $T_k(L) := \{X \in \mathcal{P}(L) \mid \tilde{H}_k(X) = 0\}$

- Lemma: Let $\chi(L)$ be the chromatic number of L. If $\dfrac{|\mathcal{F}_{k-1}(L)|}{|\mathcal{P}(L)|} < 2^{-\chi(L)-1}$, then L admits an $(k-1)$-legal system of moves, and so W_L' algebraically F_k-fibers. If moreover

  (*) $|\mathcal{F}_{\dim(L)-1}(L)| + |T_{\dim(L)}(L)| < |\mathcal{P}(L)| / 2^{\chi(L)+1}$

  then L admits a sharply $(d-1)$-legal system of moves, and so W_L' algebraically F_d-fibers with a map $W_L' \to \mathbb{Z}$ whose kernel is not of type F_{d+1}.

  Proof: Let $c: L^{(0)} \to \{1, \dots, \chi(L)\}$ be a coloring, $\mu_v := c^{-1}(c(v))$, $M = \{\mu_v \mid v \in L^{(0)}\}$
  $\Rightarrow \operatorname{rk}(\langle M \rangle) = \chi(L)$
  $\Rightarrow |\langle M \rangle| = 2^{\chi(L)}$
  $\Rightarrow \left| \mathcal{P}(L^{(0)}) / \langle M \rangle \right| = 2^{|L^{(0)}| - \chi(L)}$

Pigeonhole principle: There is a $(k-1)$-legal conf. $\boxtimes$

Obstruction: If $v \in L^{(0)}$ has degree

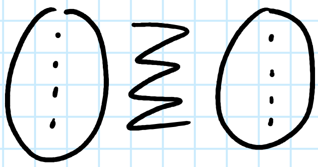
$$\frac{|\mathcal{P}(L)|}{2^{\deg(v)}} \le |\mathcal{F}_0(L)| \le \frac{|\mathcal{P}(L)|}{2^{\chi(L)+1}} \le \frac{|\mathcal{P}(L)|}{2^{\dim(L)+2}}$$

$$\deg(v) \ge \chi(L)+1 \ge \dim(L)+2$$

- Example: If L is an d-dim. simplex, then $|\mathcal{F}_d(L)|=1$ but $2^{\chi(L)+1} = 2^{d+2} > |\mathcal{P}(L)|$.

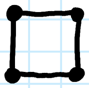
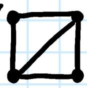
$$\frac{|\mathcal{F}_d(L)|}{|\mathcal{P}(L)|} > \frac{1}{2\cdot|\mathcal{P}(L)|} = \frac{1}{2^{\chi(L)+1}}$$

- Example: Let $K_{n,n}$ be the complete bipartite graph



If n is large enough, then $K_{n,n}$ satisfies $(*)$.

- Question: Can we choose L to be square free so that $*$ is satisfied?

 $\notin \mathcal{P}(L)$. ( $\in \mathcal{P}(L)$ is allowed)

- Computer: $\dfrac{|\mathcal{F}_0(A_2(\mathbb{F}_2))|}{|\mathcal{P}(A_2(\mathbb{F}_2))|} \approx \dfrac{2}{3}$ $\left(> 2^{-\chi(A_2(\mathbb{F}_2))-1} = \dfrac{1}{8}\right)$

- Question: How connected are random subcomplexes of finite buildings?

Part 4

Samstag, 2. Oktober 2021 00:39

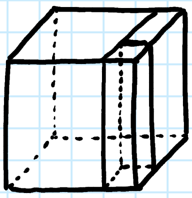
III Magic cubes

- Def: Let $n, N \in \mathbb{N}$ and let X be a set. Let $\pi_i : X^n \to X$ denote the can. projection. A measure $\mu : \mathcal{P}(X^n) \to \mathbb{R}$ is an n-dimensional magic cube of weight N if $\mu(\pi_i^{-1}(x)) = N$ for every $x \in X$.

- Example: Let $X = \{1,2,3\}$, $n = 2$, $N = 6$, and $(a_{ij}) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$. The measure $\mu : \mathcal{P}(X^2) \to \mathbb{R}$ determined by $\mu(\{(i,j)\}) = a_{ij}$ is a magic cube of weight 6:

$$\mu(\pi_1^{-1}(2)) = \mu(\{(2,1),(2,2),(2,3)\})$$
$$= a_{21} + a_{22} + a_{23} = 6.$$

- There are two canonical ways of extending the classical notion of a magic square to higher dimensions.



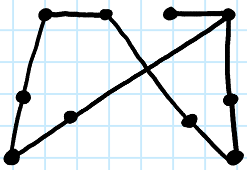
We could either sum up the entries in $1 \times 1 \times X$-blocks, which look like cylinders, or one could sum up the entries in $1 \times X \times X$-blocks.

- $\left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right)$ defines a magic cube of weight 1 that has a 0-block of side-length n.

- Lemma: Let μ be an n-dimensional magic cube over $X = \{1, \dots, t\}$. If $\mu(\{1, \dots, k\}^n) = 0$, then, $k < \frac{tn^2}{1+n^2}$. Moreover there are permutations $\sigma_i \in \mathrm{Sym}(X)$, s.t. $\mu(\{(\sigma_1(i), \dots, \sigma_n(i))\}) > 0$ for $1 \leq i \leq \lceil \frac{t}{1+n^2} \rceil$.

- Fix a finite, d-dimensional Moufang building Δ of uniform thickness $t := th(\Delta)$ and a sequence $P_1, \dots, P_n$ of distinct panels in Δ.
  Let $C_{i,1}, \dots, C_{i,t}$ be the chambers in $\mathrm{star}(P_i)$.
  For each chamber D let $x_i \in X := \{1, \dots, t\}$ be s.t. $pr_{P_i}(D) = C_{i,x_i}$.
- Lemma: Let $\Phi: Ch(\Delta) \to X^n$, $D \mapsto (x_1, \dots, x_n)$.
  The map $\mu: \mathcal{P}(X^n) \to \mathbb{N}_0$, $A \mapsto |\Phi^{-1}(A)|$ is an n-dimensional magic cube of weight $N := \frac{|Ch(\Delta)|}{t}$.
- Corollary: There are $m := \lceil \frac{t}{1+n^2} \rceil$ chambers $D_1, \dots, D_m \subseteq \Delta$ s.t. $pr_{P_i}(D_{j_1}) \neq pr_{P_i}(D_{j_2})$

  $\forall\ 1 \leq j_1 \neq j_2 \leq m$ and $1 \leq i \leq n$.



$\}D_j$

$\}P_i$

Part 5

Samstag, 2. Oktober 2021 18:22

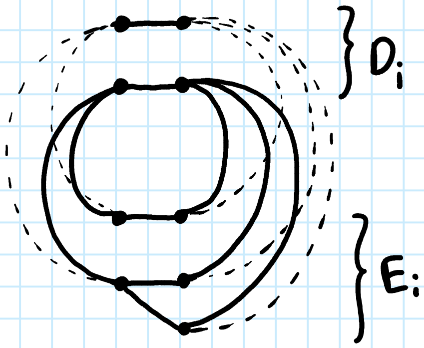
Theorem: There is a polynomial $P \in \mathbb{Z}[X,Y,Z]$, s.t.:

If $th(\Delta) > P(n,d,\ell)$, then for every n chambers $E_1,\dots,E_n \subset \Delta$ there are ℓ chambers $D_1,\dots,D_\ell \subset \Delta$, s.t.

1.) D_i and E_j are opposite $\forall i,j$

2.) $\left(\bigcup_{i=1}^{n} conv(E_i,D_j)\right) \cap \left(\bigcup_{i=1}^{n} conv(E_i,D_k)\right) = \bigcup_{i=1}^{n} E_i$

for all $1 \leq j \neq k \leq \ell$



Theorem: Let $d,k \in \mathbb{N}$, let $(\Delta_n)_{n\in\mathbb{N}}$ be a sequence of finite, uniformly thick, d-dim. Moufang buildings.
Let $t_{n,k}$ denote the set of induced subcomplexes Δ_n s.t.

1) X is a union of apartments in Δ_n
2) X is a chamber complex
3) Every subcomplex $Y \subseteq X$ with at most k vertices is contained in a d-spherical subcomplex $Z \subseteq X$. ($Z \simeq \bigvee_{i\in I} S^d$) that is a union of at most k apartments.

If $th(\Delta_n) \geq n$ $\forall n$ then $\forall \delta > 0$

$$\frac{|t_n|}{|\mathcal{P}(\Delta_n)|} > 1-\delta^n \text{ for almost every } n.$$

If $th(\Delta_n) \to \infty$, then $\frac{|t_n|}{|\mathcal{P}(\Delta_n)|} \to 1.$

- Corollary: If $d \geq 2$ then $\frac{|\mathcal{F}_1(\Delta_n)|}{|\mathcal{P}(\Delta_n)|} \to 0, n \to \infty$.

Proof:

$\boxtimes$

- Fix a sequence $(p_n)_{n\in\mathbb{N}}$ of ascending primes and let $\Delta_{k,n} := A(\mathbb{F}_{p_n}^{k+1})$ (type A_k).

- Theorem: For every $k \geq 2$ there is some $\varepsilon \in (0,1)$ with

$$\frac{|\mathcal{F}_{\lfloor\frac{k-1}{2}\rfloor}(\Delta_{k,n})| + |T_{k-1}(\Delta_{k,n})|}{|\mathcal{P}(\Delta_{k,n})|} < \varepsilon^n \quad \text{for almost every } n.$$

- Corollary: For every $\varepsilon > 0$ there is a finite, bipartite, square free graph Υ, s.t.

$$|\mathcal{F}_0(\Upsilon)| < \varepsilon |\mathcal{P}(\Upsilon)|.$$

- Υ can be chosen to have girth$(\Upsilon) = 16$ (Building of type $I_2(8)$).

- Question: Given $g \in \mathbb{N}$, $\varepsilon > 0$. Is there a finite graph Υ, s.t. girth$(\Upsilon) \geq g$, $c(\Upsilon) \geq 1 - \varepsilon$?

- Conjecture: Let $d \in \mathbb{N}$ and let $(\Delta_n)_{n\in\mathbb{N}}$ be a sequence of d-dimensional buildings with $\mathrm{th}(\Delta_n) \to \infty$. Then $\frac{|\mathcal{F}_{d-1}(\Delta_n)| + |T_d(\Delta_n)|}{|\mathcal{P}(\Delta_n)|} \to 0, n \to \infty$.

Part 6

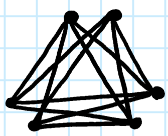
Samstag, 2. Oktober 2021 20:07

VI Applications

- Theorem: If $k \geq 2$ then $W'_{\Delta_{k,n}}$ algebraically $F_{\lfloor \frac{k+1}{2} \rfloor}$-fibers for almost every n. If moreover $k \leq 3$ then there is some $\chi : W'_{\Delta_{k,n}} \twoheadrightarrow \mathbb{Z}$, s.t. $\ker(\chi)$ is of type F_{k-1} but not F_k.

- Conjecture: Let $d \in \mathbb{N}$ and let $(\Delta_n)_{n\in\mathbb{N}}$ be a sequence of d-dimensional buildings with $th(\Delta_n) \to \infty$. Then there is a map $\chi : W'_{\Delta_n} \to \mathbb{Z}$, s.t. $\ker(\chi)$ is of type F_d but not F_{d+1}.

- Lemma: Let m be maximal with $\mathbb{Z}^m \hookrightarrow W_{A_n(p)}$. Then $m = \lfloor \frac{n}{2} \rfloor + 1$.

- Proof:  $\rightsquigarrow$  $\rightsquigarrow D_\infty^3 \geq \mathbb{Z}^3$

  1.) m is maximal with $\overset{m}{\underset{i=1}{\times}} \overbrace{\{v_i, w_i\}}^{D_\infty} \leq A_n(p)$

  2.) This is the only source for free abelian subgroups in Coxeter groups. (Well-known result of Daan Krammer.)

- N. Brady (2000): Let $n \geq 3$. Is there a group G of type F_n but not F_{n+1} with $\mathbb{Z}^{n-1} \not\leq G$?
- R. Kropholler (2018): Yes.