The type I conjecture on trees

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Unitary representations

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Why do we study unitary representations ?

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Groups G

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This define a continuous **faithfull** action of G on a Hilbert space.

Unitary representations of locally compact groups

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Representations

A **unitary representation** of a locally compact group G is a continuous group homomorphism :

$$\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$$

where $\mathcal{H}_\pi$ is a complex Hilbert space.

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Two representations π and σ are said to be **equivalent** when there exists a unitary operator :

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such that

$$\mathcal{U}\pi(g) = \sigma(g)\mathcal{U} \quad \forall g \in G.$$

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Irreducible representations

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where $\mathcal{H}_\pi$ is a complex Hilbert space.

Irreducible representations

A representation is called **irreducible** when the only **closed** invariant subspaces $W \subseteq \mathcal{H}_\pi$ (e.g $\pi(G)W \subseteq W$) are $\{0\}$ and $\mathcal{H}_\pi$.

Decomposition of representations

Decomposition of representations

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Let G be a **finite** group.

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Let G be a **finite** group.
Every representation π of G **decomposes uniquely** as a **direct sum** of **irreducible** representations of G :

$$\pi \simeq \bigoplus_{i \in \mathbb{N}} n_i \sigma_i .$$

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Let G be a **compact** group.
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Remark

The study of representations of **compact** groups reduces to the study of **irreducible** representations.

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Let G be a **locally compact** group.
Every representation π of G **decomposes (possibly not uniquely)** as a **direct integral** of **irreducible** representations of G :

$$\pi \simeq \int_{\widehat{G}} \sigma \, \mathrm{d}\nu_\pi(\sigma).$$

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Examples

Let σ be an **irreducible** representation. The representation:

$$\pi \simeq n\sigma$$

is a **factorial** representation for every $n = \infty, 1, 2, \ldots$

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Let F_2 be the free group with two generators.

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Examples

Let F_2 be the free group with two generators. The regular representation

$$(\lambda_{F_2}, L^2(F_2, \mu))$$

is a **factorial** representation.

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Let σ and τ be **not equivalent irreducible** representations.

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Non-example

Let σ and τ be **not equivalent irreducible** representations. The representation:

$$\pi \simeq n_1\sigma \oplus n_2\tau$$

is a **not factorial** for any $n_1, n_2 \geq 1$.

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Every representation π of G **decomposes uniquely** as a **direct integral** of **factorial** representations of G :

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Type I groups

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A locally compact group is called a **type I** group when every of its factorial representations is of the form:

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for some irreducible representation σ and $n = \infty, 1, 2, \ldots$

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For **type I** groups

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Remark

For **type I** groups the decomposition of representations into **direct integral** of **irreducible** representations is **unique**.

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Examples of type I groups

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- **Free** groups with rank ≥ 2.

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Type I conjecture on trees

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Which closed subgroup of $\mathrm{Aut}(T)$ is **type I**

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Which closed subgroup of $\mathrm{Aut}(T)$ is **type I** amoung **non-amenable** groups ?

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- G fixes a **point in the boundary**.
- G stabilizes a **pair of points in the boundary**.

The type I conjecture on trees

Type I conjecture on trees (Nebbia-Houdayer-Raum)

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Let T be a locally finite thick tree and let $G \leq \mathrm{Aut}(T)$ be a closed non-amenable subgroup acting minimaly on T.

The type I conjecture on trees

Type I conjecture on trees (Nebbia-Houdayer-Raum)

Let T be a locally finite thick tree and let $G \leq \mathrm{Aut}(T)$ be a closed non-amenable subgroup acting minimaly on T.
Then, G acts **transitively** on the boundary if and only if it is **type I**.

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Let Δ be a locally finite thick building of type $I_2(\infty)$ and let $G \leq \mathrm{Aut}(\Delta)$ be a closed non-amenable subgroup acting minimaly on Δ.

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Let Δ be a locally finite thick building of type $I_2(\infty)$ and let $G \leq \text{Aut}(\Delta)$ be a closed non-amenable subgroup acting minimaly on Δ.
Then, G comes from a **B-N** pair if and only if it is **type I**.

Part I

Part I : G acts transitively on the boundary then G is type I.

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Partially solved

CCR groups

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into irreducibles of K

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Criterion for "type I"ness

CCR groups are **type I**.

CCR groups

Examples of CCR groups

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- Compact groups.
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Representations of $\mathrm{Aut}(T)$

The topology of $G \leq \mathrm{Aut}(T)$

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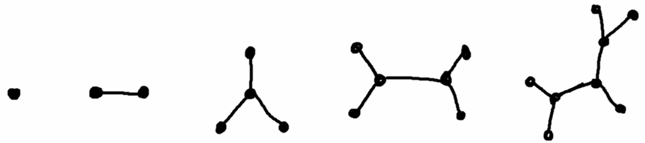
$$\mathrm{Fix}_{\mathrm{Aut}(T)}(\mathcal{T}) = \{g \in \mathrm{Aut}(T) | \forall v \in \mathcal{T} : gv = v\}$$

where $\mathcal{T}$ is a complete finite subtree



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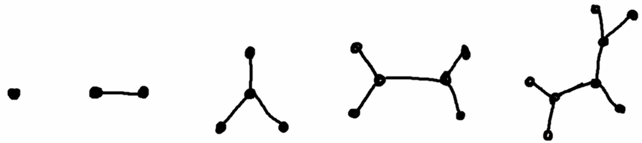
$$\mathrm{Fix}_{\mathrm{Aut}(T)}(\mathcal{T}) = \{g \in \mathrm{Aut}(T) | \forall v \in \mathcal{T} : gv = v\}$$

where $\mathcal{T}$ is a complete finite subtree form a **b.o.n.c.o.** of G.



The irreducible representations of $\mathrm{Aut}(T)$

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Lemma

Let π be a representation of G.

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Lemma

Let π be a representation of G. Then, there **exists** a complete finite subtree $\mathcal{T}$ of T such that π admits a **non-zero** $\mathrm{Fix}_G(\mathcal{T})$**-invariant vector** $\xi \in \mathcal{H}_\pi$ i.e. $\pi(k)\xi = \xi \ \forall k \in \mathrm{Fix}_G(\mathcal{T})$.

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Proof.

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Proof.

Let $\xi \in \mathcal{H}_\pi$ be non trivial.

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Proof.

Let $\xi \in \mathcal{H}_\pi$ be non trivial. Now, $\int_{\mathrm{Fix}_G(\mathcal{T})} \pi(g)\xi \mathrm{d}\mu(g)$ is $\mathrm{Fix}_G(\mathcal{T})$-invariant.

$\square$

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Proof.

Let $\xi \in \mathcal{H}_\pi$ be non trivial. Now, $\int_{\mathrm{Fix}_G(\mathcal{T})} \pi(g)\xi \mathrm{d}\mu(g)$ is $\mathrm{Fix}_G(\mathcal{T})$-invariant.
The function $g \mapsto < \pi(g)\xi, \xi >$ is continuous and non-zero at $g = 1_G$.
$\exists$ a big enough $\mathcal{T}$ such that $\int_{\mathrm{Fix}_G(\mathcal{T})} \pi(g)\xi \mathrm{d}\mu(g) \neq 0$. □

Minimal subtree

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Given a representation π of G, what is the **smallest** complete finite subtree $\mathcal{T}$ for which it admits invariant vectors ?

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Three types

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- **an edge**

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- **bigger than an edge**

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- **bigger than an edge** π is called **cuspidal**.

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If the minimal subtree $\mathcal{T}$ is :

- **a vertex** π is called **spherical**.
  G is transitive on the boundary
- **an edge** π is called **special**.
  G is transitively on the boundary
- **bigger than an edge** π is called **cuspidal**.

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This **solves** the **part I of the type I conjecture** for numerous semi-regular trees such as the $(34, 55)$-semiregular tree.

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Let T be a locally finite thick tree and let $G \leq \mathrm{Aut}(T)$ be a **Type I** closed non-amenable subgroup acting minimaly on T.

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Let T be a locally finite thick tree and let $G \leq \mathrm{Aut}(T)$ be a **Type I** closed non-amenable subgroup acting minimaly on T. Then, G has a cocompact amenable subgroup and hence acts transitively on the boundary of T.

Theorem

Let $G \leq \mathrm{Aut}(T)$ be a **Type I** closed non-amenable subgroup acting minimaly on T. Then, G acts **transitively** on the boundary of the tree.

Thank you

Thank you !!