Divergence in Coxeter groups

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Outline

This is joint work with Pallavi Dani, Yusra Naqvi, Dibyendu Roy and Ignat Soroko.

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2. Right-angled Coxeter groups

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3. General Coxeter groups

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2. Right-angled Coxeter groups
3. General Coxeter groups (infinite and non-affine)

Divergence of geodesics

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Definition (Gromov 1980s)

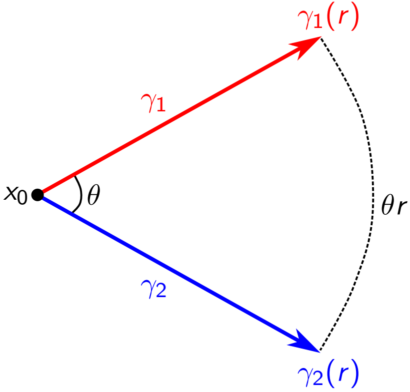
The divergence of γ_1 and γ_2 at time r is

$$\operatorname{div}(\gamma_1, \gamma_2, r) := \inf_p \operatorname{length}(p)$$

where the infimum is taken over all rectifiable paths p in $X \setminus \operatorname{Ball}(x_0, r)$ connecting $\gamma_1(r)$ and $\gamma_2(r)$.

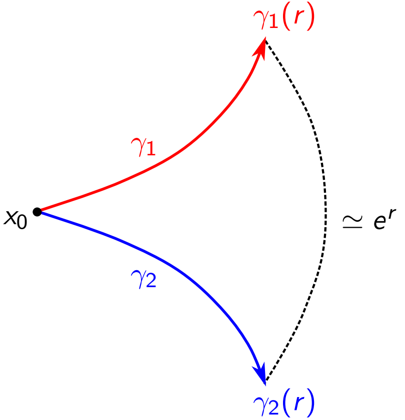
Divergence of geodesics in Euclidean space

In Euclidean space, all pairs of geodesics diverge linearly.



Divergence of geodesics in hyperbolic space

In hyperbolic space, all pairs of geodesics diverge exponentially.



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Let X be a symmetric space of noncompact type e.g. $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$. Then for all pairs of geodesics γ_1, γ_2 with common basepoint, the function $r \mapsto \mathrm{div}(\gamma_1, \gamma_2, r)$ is either linear or exponential.

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It doesn't.

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Definition (Gersten 1994)

The divergence of G is the function

$$\mathrm{div}_G(r) := \sup_{x,y} \left(\inf_p \mathrm{length}(p) \right)$$

where

- the sup is over all pairs of points $x, y \in X$ at distance r from e
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G has linear divergence if $\mathrm{div}_G(r) \simeq r$, quadratic divergence if $\mathrm{div}_G(r) \simeq r^2$, etc, where

$$f \preceq g \iff \exists C > 0 \text{ s.t. } f(r) \leq Cg(Cr + C) + Cr + C$$

These rates of divergence are quasi-isometry invariants (Gersten).

Previous results on divergence

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- lattices in higher rank semisimple Lie groups conjectured to have linear divergence; proved in some cases e.g. $\mathrm{SL}(n, \mathbb{Z})$ [Drutu–Mozes–Sapir 2010]
- right-angled Artin groups have divergence linear, quadratic or exponential [Abrams–Brady–Dani–Duchin–Young 2010, Behrstock–Charney 2012]

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Constructions of finitely presented CAT(0) groups with divergence:

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- $r^n \log r$ for all integers $n \geq 1$ [Brady–Tran 2020]

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The right-angled Coxeter group (RACG) associated to Γ is

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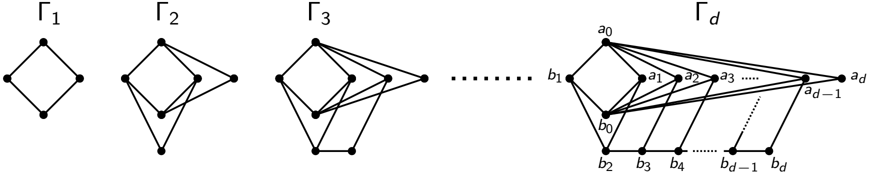
Theorem 1 (Dani–T 2015)

1. *W_Γ has linear divergence $\iff$ Γ a join $\iff$ W_Γ reducible.*
2. *W_Γ has quadratic divergence if and only if Γ is $\mathcal{CFS}$ and is not a join.*

Divergence in right-angled Coxeter groups

Theorem 2 (Dani–T 2015)

For all $d \geq 1$, the group W_{Γ_d} has divergence r^d.



Subsequent results for RACGs

Behrstock, Falgas-Ravry, Hagen and Susse generalised our $\mathcal{CFS}$ condition to all Γ.

Theorem (Behrstock–Caprace–Hagen–Sisto, Sisto, Levcovitz)

1. *The divergence of W_Γ is either exponential ($\iff$ the group is relatively hyperbolic) or bounded above by a polynomial ($\iff$ the group is thick).*
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Theorem (Levcovitz 2020)

W_Γ has hypergraph index h $\iff$ W_Γ has divergence r^{h+1}.

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Proof ingredients:

- General result of Kapovich–Leeb for locally compact CAT(0) spaces
- For W other than in (1) and (2), and $w \in W$ a Coxeter element, w^∞ is a rank one geodesic (Caprace–Fujiwara)

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For $i \geq 0$

- for $T, T' \in \Lambda_i$, define $T \sim T'$ if $\exists$

$$T = T_0, T_1, \ldots, T_m = T' \subset \Lambda_i$$

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Define hypergraph index $h(W, S) = i$ if $S \in \Lambda_i \setminus \Lambda_{i-1}$, else $h(W, S) = \infty$.

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Conjecture

If $1 \leq h < \infty$ then W has divergence bounded below by r^{h+1}.

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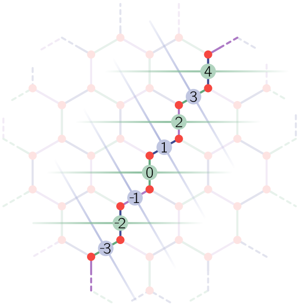
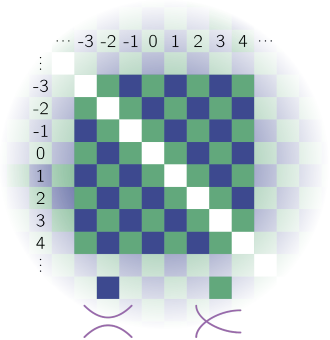
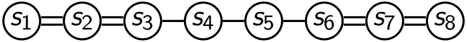
- Use root system to determine intersections of walls

Computation in type $\tilde{A}_2$ ($h = 0$)





Computations for $h = 1$



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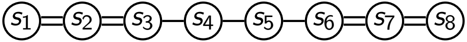

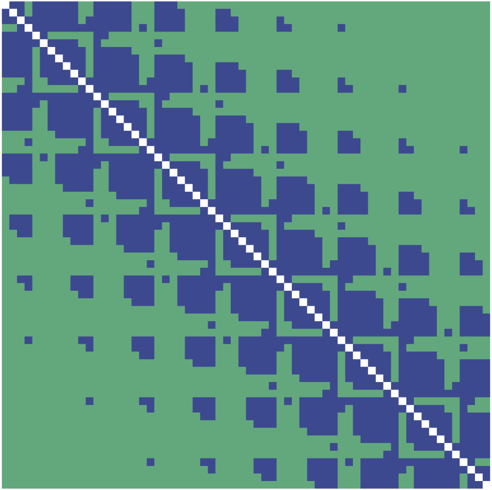

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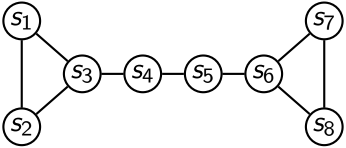

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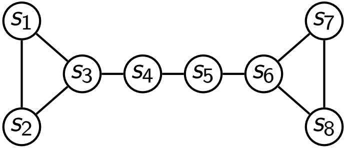

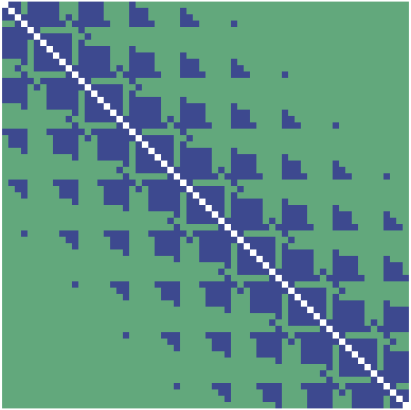

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