Strong transitivity, Moufang's condition and the Howe-Moore property

- Buildings 2021 -

Motivation

- Well known families of locally compact groups:
  1) Connected, non-compact, simple real Lie groups with finite center (e.g. $SL(n,\mathbb{R})$)
  2) Isotropic simple algebraic groups over non-Archimedean local fields (e.g. $SL(n,\mathbb{Q}_p)$)
  3) Closed, topologically simple subgroups of $Aut(T)$ with a 2-transitive action on the boundary of the bi-regular tree T, that has valence ≥ 3 at every vertex (e.g. the universal group of Burger-Mozes $U(F)^+$, when F is 2-transitive.

Similarities:
- all those groups act on their associated "symmetric spaces."
- their action is "strongly transitive" ($\forall (A_i, c_i)$, $c_i \in Ch(A_i)$ $\exists g \in G$ s.th. $g(A_1) = A_2$, $g(c_1) = c_2$). (2-tran for $T \Leftrightarrow$ str. tran.)
- KAK, KAN, BWB decompositions
- the Howe-Moore property

(Open) Questions about the Howe-Moore property

- The Howe-Moore property of a l.c.g G is a harmonic analytic prop regarding all unitary representations of G; H-M $\Rightarrow$ strongly mixing for $H \leq G \curvearrowright (X,\mu) \Rightarrow$
  $\Rightarrow$ ergodicity for $H \leq G \curvearrowright (X,\mu)$
- Used in Ratner's theorems, Mostow rigidity, equidistribution results...

Question 1: Let Δ a locally finite thick affine building. If $H \leq \mathrm{Aut}(\Delta)$ acts str. tran does H have the H-M property? This is true when $\Delta = T$ bi-reg tree.

Question 2: If $H \leq \mathrm{Aut}(\Delta)$ has the H-M prop, does H act strongly transitively on Δ?

Question 3: What are other examples of l.c.g. with the H-M property, or that act strongly transitively on Δ?

Conjecture: For groups acting on buildings, only the top. simple, closed, str. tran and type-pres. groups acting on affine Δ can have the H-M.

Some partial answers for $H = U(F)^+ \leq \mathrm{Aut}(T_d)$: the Howe-Moore property $\Leftrightarrow F$ is 2-tran.

Some answers to Q.3

What are other examples of l.c.g. that act strongly transitively on Δ?

Theorem 1 (C. '21) Let Δ be a locally finite thick affine building of dim ≥ 2. Let H be a closed subgp of $\mathrm{Aut}(\Delta)$ that acts strongly transitively and type-preservingly on Δ. Then H is Moufang (i.e. for every root α of $\partial_\infty\Delta$, the "root" group of H
$U_\alpha(H) := \{ g \in H \mid g \text{ fixes pointwise every chamber } c \in Ch(\partial_\infty \Delta) \text{ having a panel in } \alpha - \partial\alpha \}$
acts transitively on the set $\mathcal{A}_{\partial_\infty\Delta}(\alpha)$ of all apartments in $\partial_\infty\Delta$ having α as half apart).

Theorem 2 (C. '21). Let Δ be an irreducible, locally finite thick affine building of dim ≥ 2. s.th. $\mathrm{Aut}_0(\Delta)$ acts strongly transitively on Δ. Then the only topologically simple, closed and strongly transitive subgroup H of $\mathrm{Aut}_0(\Delta)$ is the $G^+ = \mathbb{G}(k)^+$ of the isotropic simple algebraic group $\mathbb{G}$ over a non-Archimedean local field k associated with Δ.

$G^+ := \langle U_\alpha(G) \mid \alpha \text{ a root of } \partial_\infty\Delta \rangle$

Remark about thm 2: By an isotropic simple alg group G over a non-Archimedean local field k, we mean a semisimple, (absolutely) almost simple algebraic group over k, of k-rank ≥ 1. If G is simply connected, by Borel-Tits 73, $G(k) = G(k)^{+} := \langle U_\alpha(G(k)) \mid \alpha \text{ a root of } \partial_\infty \Delta \rangle$.

Following Weiss (The structure of affine buildings) and a remark in Caprace-Monod 2015:

Theorem (non-trivial): Let Δ be an irreducible locally finite thick affine building of dim ≥ 2 whose building at infinity $\partial_\infty \Delta$ is Moufang. Then Δ is precisely a Bruhat-Tits building associated with the group $G(k)$ of an isotropic simple algebraic group G over a non-Archimedean local field k, and of k-rank ≥ 2.

Question 3: What are other examples of l.c.g. that act strongly transitively on Δ?

Answer: Thm 1 + Thm 2 $\Rightarrow$ the only top. simple, closed and str. tran. $\leq \mathrm{Aut}_0(\Delta)$ are of algebraic origin, where Δ is irr., loc. finite, thick affine.

Ideas to prove Thm 1

Theorem 1 (C. '21) Let Δ be a locally finite thick affine building of dim ≥ 2. Let H be a closed subgp of $\mathrm{Aut}(\Delta)$ that acts strongly transitively and type-preservingly on Δ. Then H is Moufang (i.e. for every root α of $\partial_\infty \Delta$, the "root" group of H $U_\alpha(H) := \{ g \in H \mid g \text{ fixes pointwise every chamber } c \in Ch(\partial_\infty \Delta) \text{ having a panel in } \alpha - \partial\alpha \}$ acts transitively on the set $\mathcal{A}_{\partial_\infty \Delta}(\alpha)$ of all apartments in $\partial_\infty \Delta$ having α as half apart).

- The proof is the same as in Caprace-Monod 2015, where the case $H = \mathrm{Aut}_0(\Delta)$ is treated: Theorem (CM): Let $\partial_\infty \Delta$ be a thick irreducible spherical building of dim ≥ 1. If $\mathrm{Isom}(\Delta)_\xi$ acts cocompactly on Δ, $\forall\, \xi \in \partial_\infty \Delta$, then $\partial_\infty \Delta$ is Moufang.
- Fix a root α of $\partial_\infty \Delta$. Then as H is strongly transitive on Δ (and so on $\partial_\infty \Delta$)

$\Rightarrow H_\alpha := \{ g \in H \mid g(\alpha) = \alpha \text{ pointwise} \}$ is transitive on $\mathcal{A}_{\partial_\infty \Delta}(\alpha)$.

- It is enough to prove $H_\alpha \cap U_\alpha(H)$ is still transitive on $\mathcal{A}_{\partial_\infty\Delta}(\alpha)$
- Take $P_1, \dots, P_N$ the panels of $\alpha - \partial\alpha$. Then $H_\alpha \leq \bigcap_{i=1}^{N} H_{P_i}$
- For each P_i we have the Levi decomposition: $H_{P_i} = H_{P_i, P_i^-} \cdot H_{P_i}^U$
  where $H_{P_i}^U$ is the "unipotent radical" of H_{P_i}
- $H_{P_i}^U$ fixes pointwise every chamber of $\partial_\infty\Delta$ having P_i as a panel
- $H_{P_i}^U$ acts transitively on $Opp(P_i) = \{$the panels in $\partial_\infty\Delta$ opposite $P_i\}$.
- Take $V_0 = H_\alpha$ and for each $i \in \{1, \dots, N\}$ take $V_i := V_{i-1} \cap H_{P_i}^U$
- By using the Levi decomposition of subgroups $N \leq H_\xi$ (with some properties)

$$\begin{cases} N = N_{\xi_-} \cdot N^U \quad ; \quad N^U := N \cap H_\xi^U \text{ - "unipotent radical"} \\ N^U \text{ - acts transitively on the } N\text{-orbits of } \xi_- \text{ in } Opp(\xi). \end{cases}$$

  one can prove by induction on $i \in \{0, \dots, N\}$ that V_i acts transitively on $\mathcal{A}_{\partial_\infty\Delta}(\alpha)$. $\quad V_{i-1} = V_{i-1, P_i^-} \cdot V_i$, V_i is the "unipotent radical" of V_{i-1}

Ideas to prove Thm 2

<u>Theorem 2</u> (C. '21). Let Δ be an irreducible, locally finite thick affine building of dim ≥ 2. s.th. $Aut_0(\Delta)$ acts strongly transitively on Δ. <u>Then</u> the only topologically simple, closed and strongly transitive subgroup H of $Aut_0(\Delta)$ is the $G^+ := \mathbb{G}(k)^+$ of the isotropic simple algebraic group $\mathbb{G}$ over a non-Archimedean local field k associated with Δ.

$$G^+ := \langle U_\alpha(G) \mid \alpha \text{ a root of } \partial_\infty\Delta \rangle$$

- $U_\alpha(Aut_0(\Delta)) = U_\alpha(H) = U_\alpha(\mathbb{G}(k))$ as they act simple-transitively on $\mathcal{A}_{\partial_\infty\Delta}(\alpha)$
  $\forall\ \alpha$ a root of $\partial_\infty\Delta$
- $U^\pm := \langle U_\alpha(Aut_0(\Delta)) \mid \alpha \in \Phi^\pm \rangle$, then $\{id\} \neq \overline{\langle U^+, U^- \rangle}$ normal in H, resp. $\mathbb{G}(k)$
- $\overline{\langle U^+, U^- \rangle} = H^+$
- $\mathbb{G}(k)^+ = \overline{\langle U^+, U^- \rangle} \xRightarrow[\text{simple}]{H \text{ top}} H = \mathbb{G}(k)^+$.

Some answers to Q1 and Q3

Theorem 3 (C. '20) Let Δ be a locally finite thick affine building and G a topologically simple, closed, strongly transitive and type-preserving subgroup of $Aut(\Delta)$. Then G has the Howe-Moore property.

Remark about Thm 3:
- The proof of Thm 3 is different than the strategy used so far in the literature, i.e. the KA^+K decomposition and A^+ is abelian
- the proof is a generalization of the proof of the H-M property given in Burger-Mozes 2000 for the case of bi-regular trees.
- A Bruhat-like decomposition "LHL" where $L = G^0_{St(\sigma, \partial\infty)}$ and H a subset of G, $L \leq B^0$, B = Borel

By Thm.1 and Thm 2, no new examples of groups with the H-M property ☹

About the Conjecture

Conjecture: For groups acting on general buildings, only the topologically simple, closed, strongly transitive and type-preserving groups acting on affine buildings can have the Howe-Moore property.

- Need to construct "new" classes of unitary representations, in particular some that do not have G-invariant non-zero vectors and with some matrix coefficients not vanishing at infinity.
- For the universal group $U(F)^+ \leq Aut(T_d)$ of Burger-Mozes I have proved that $U(F)^+$ has the Howe-Moore property if and only if F is 2-transitive ($\Leftrightarrow$ str. tran)
- I have constructed a "non-standard" Hilbert space and "non-standard" unitary representation without $U(F)^+$-invariant vectors and with non-vanishing at ∞ matrix coefficients.

Thank you!