The boundary rigidity of lattices in products of trees

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joint work with

Kasia Jankiewicz, Kim Ruane and Bakul Sathaye

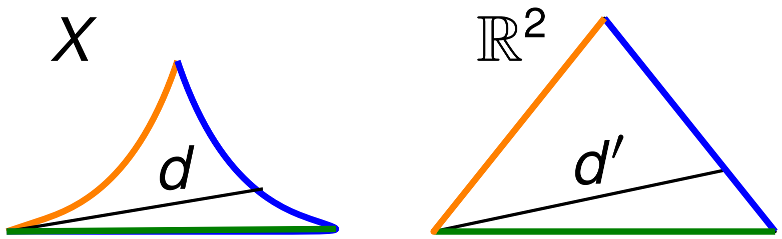
Outline

Goal: Show boundary rigidity of lattices in products of trees

- CAT(0) spaces and their boundaries
- Lattices in products of trees
- Theorem with proof idea

CAT(0) spaces

All spaces in this talk are CAT(0) (if not said otherwise)



$$d \leq d'$$

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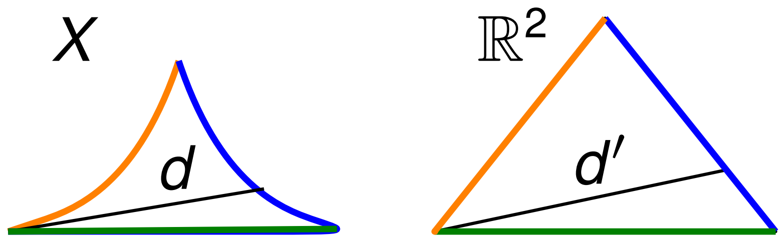

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Examples:

- $\mathbb{R}^2$, $\mathbb{H}^2$, trees,

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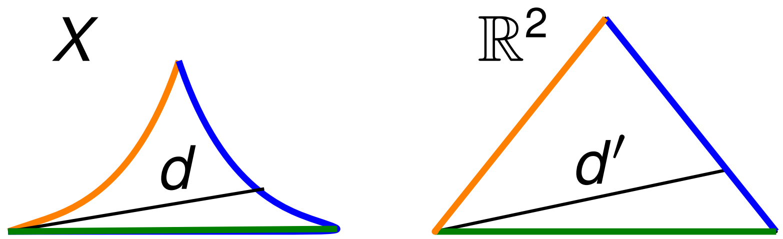

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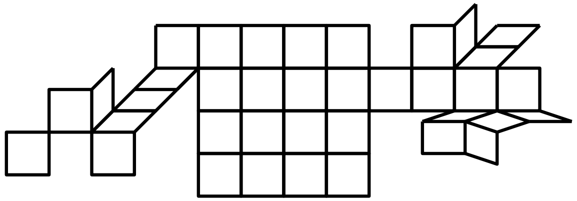

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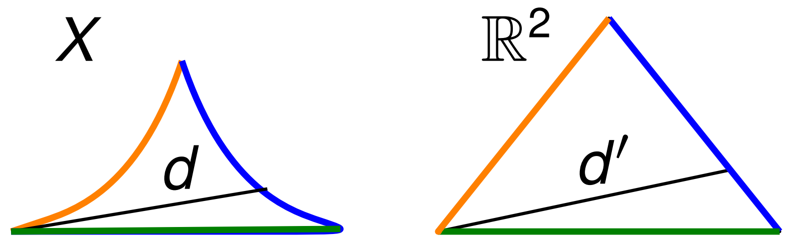

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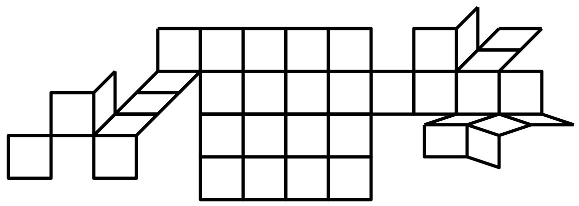

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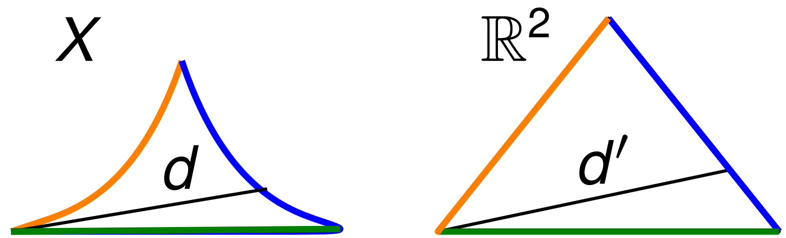

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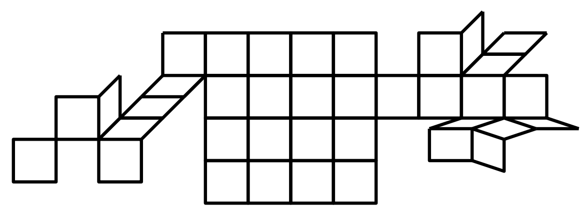

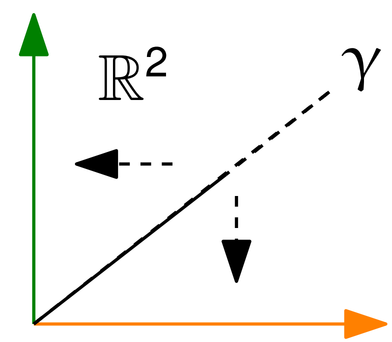

daim(X_1) = diam(X_2) = $\infty \Rightarrow$ each geodesic ray γ lies in a flat!

CAT(0) spaces

Example: T_n, T_m regular trees

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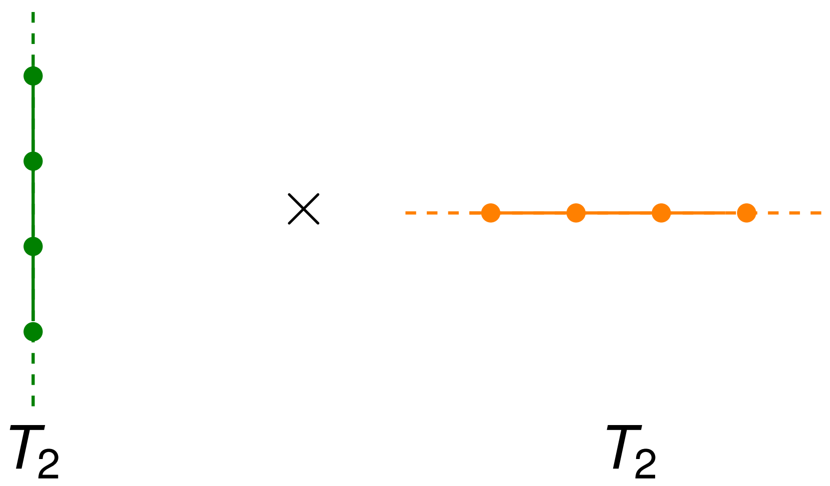

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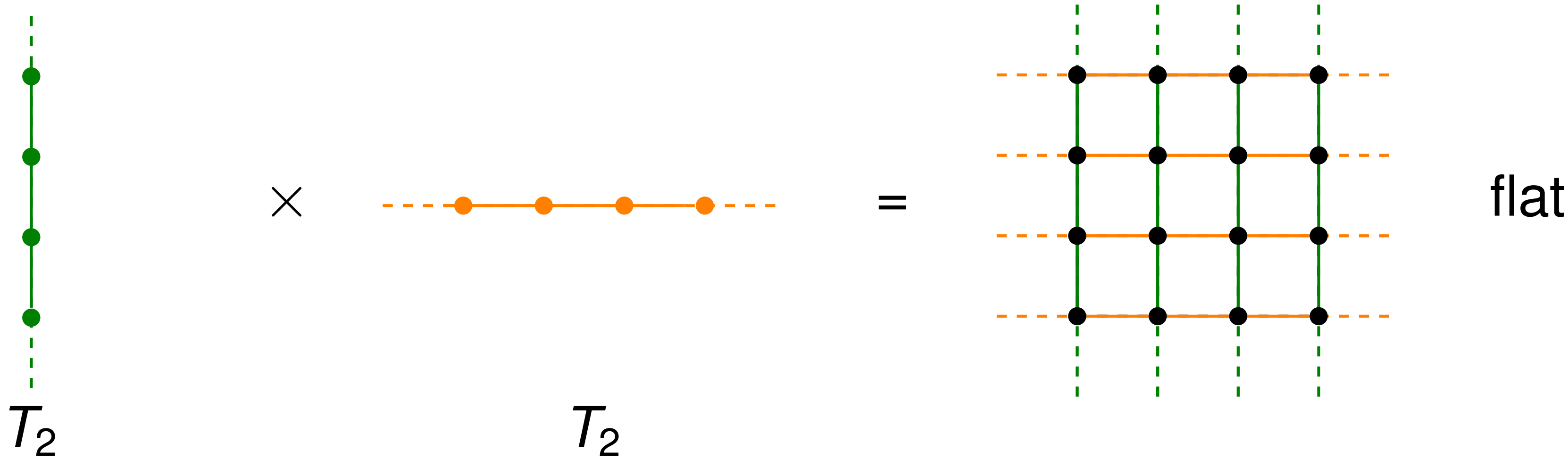

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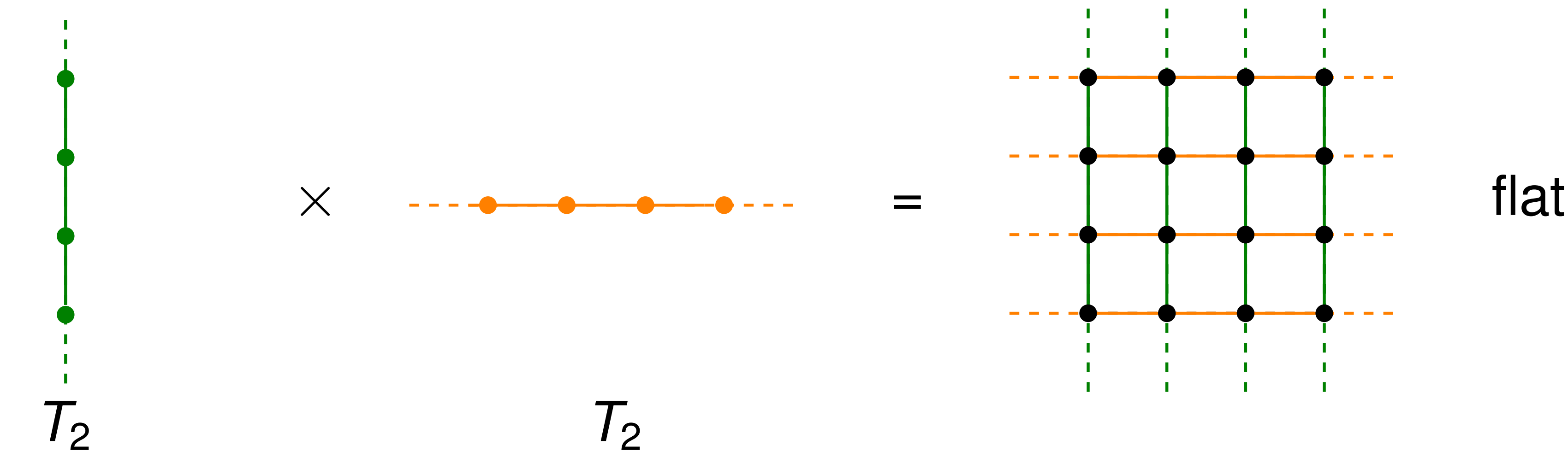

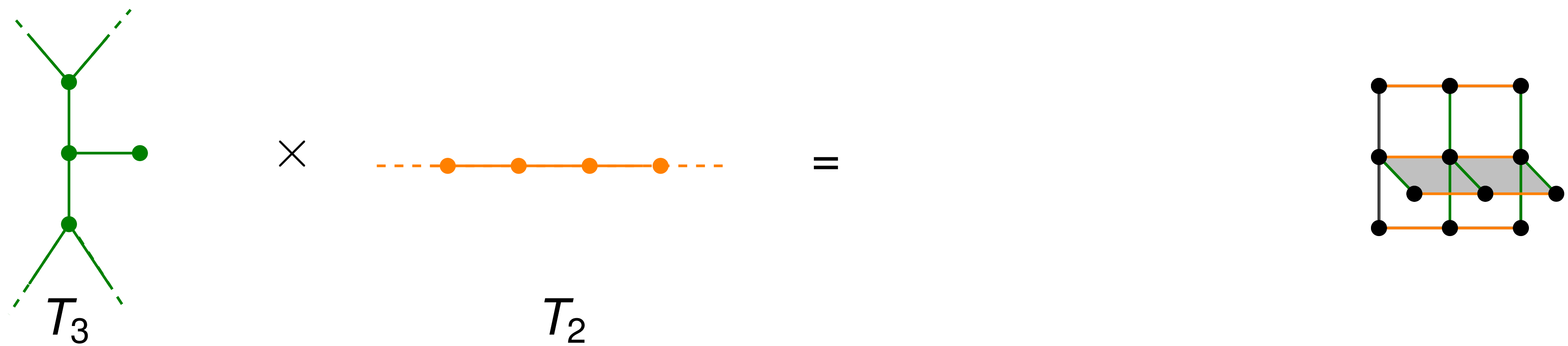

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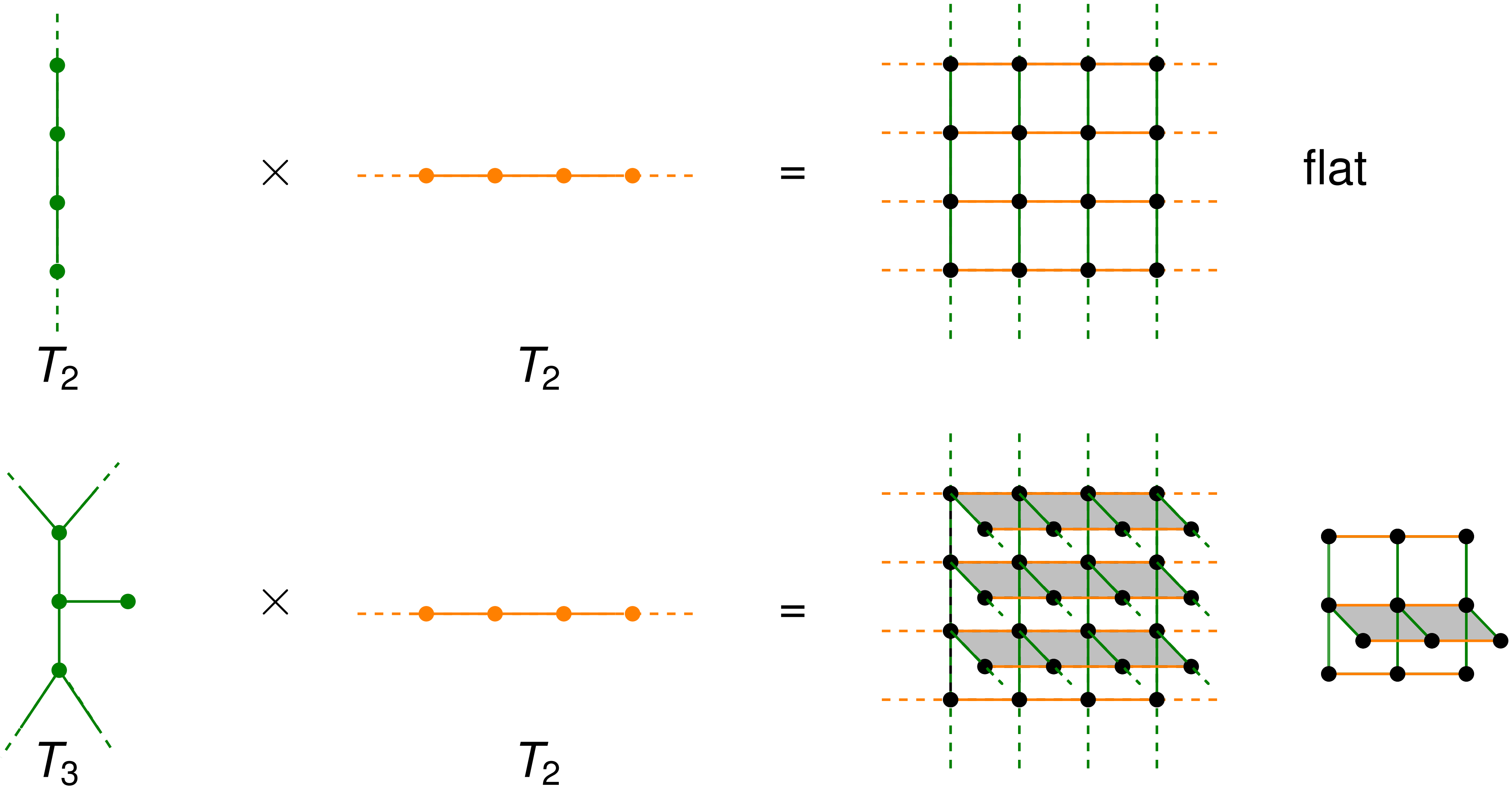
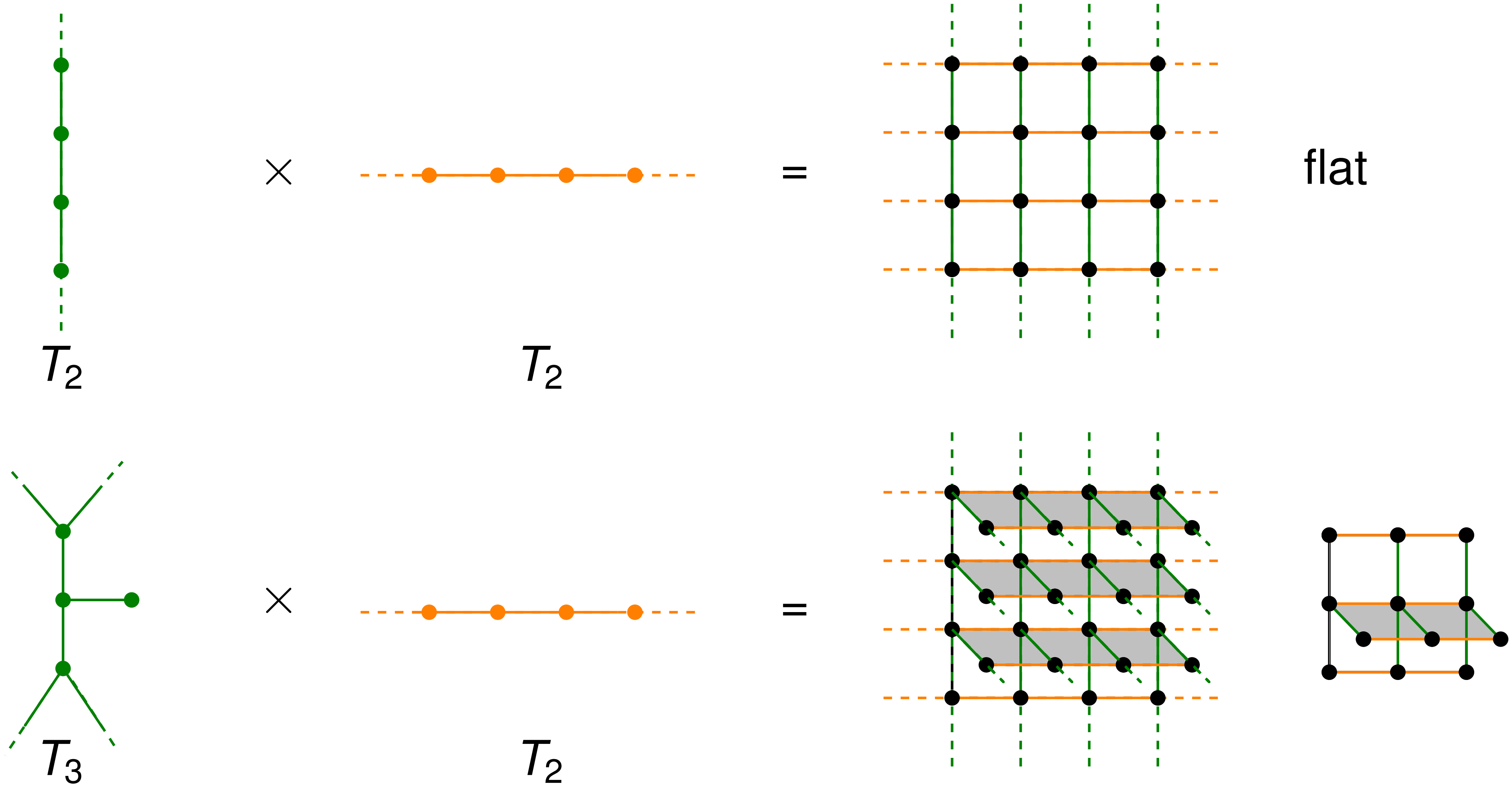


CAT(0) spaces

Example: T_n, T_m regular trees $\Rightarrow$ $T_n \times T_m$ CAT(0) square complex



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$\gamma_1 \sim \gamma_2$ iff γ_1,γ_2 are close, i.e. $\exists C > 0$ s.t. $d(\gamma_1(t), \gamma_2(t)) \leq C \; \forall t \geq 0$.

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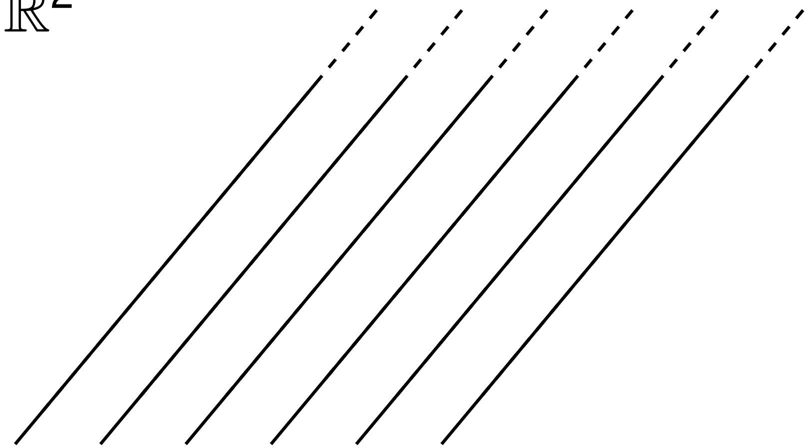

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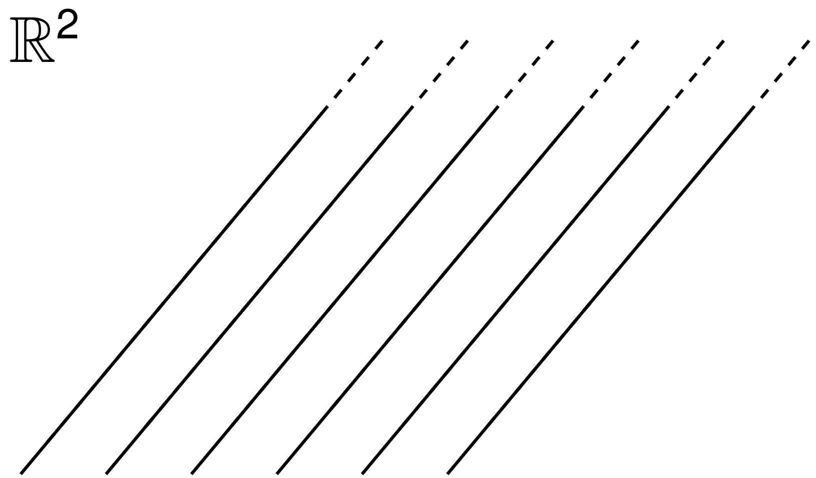

Fact: $\forall\ p \in X, \bar{\gamma} \in \partial X\ \exists$ a geodesic ray $\gamma_p \in \bar{\gamma}$ that starts at p.

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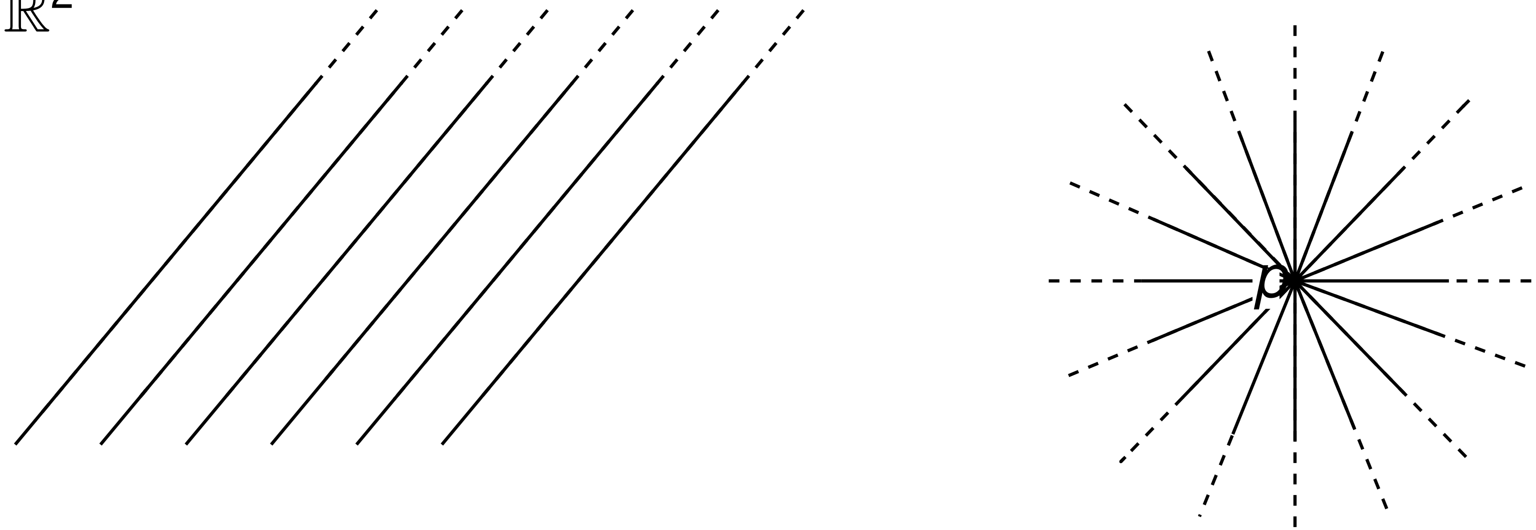

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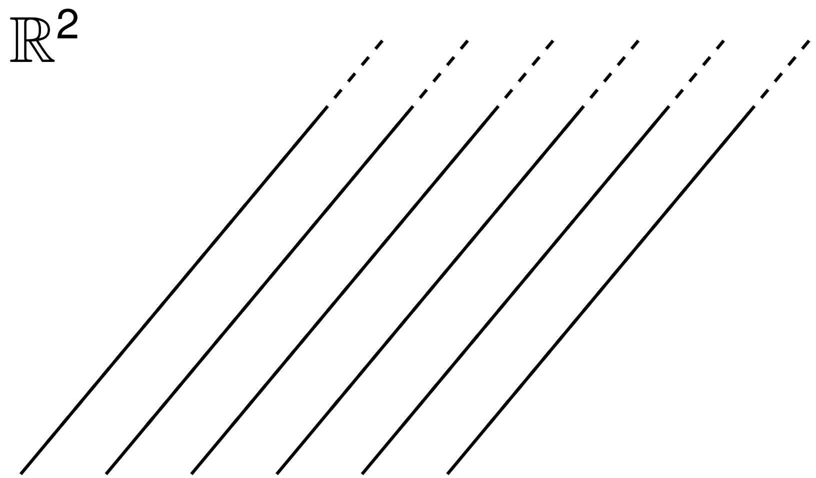

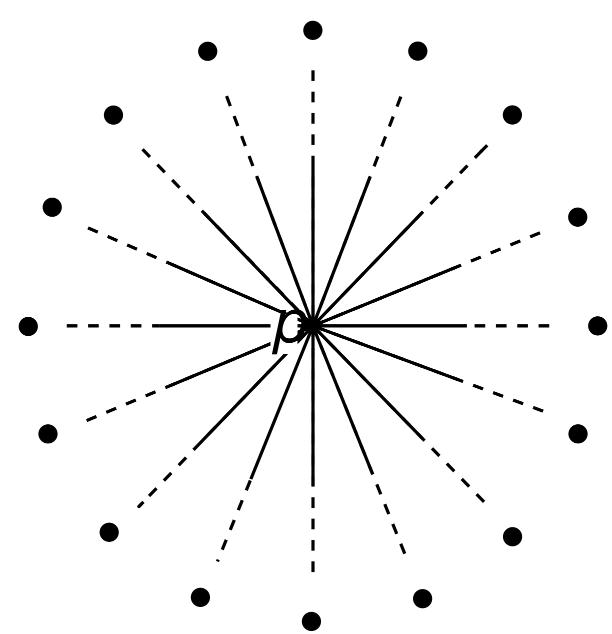

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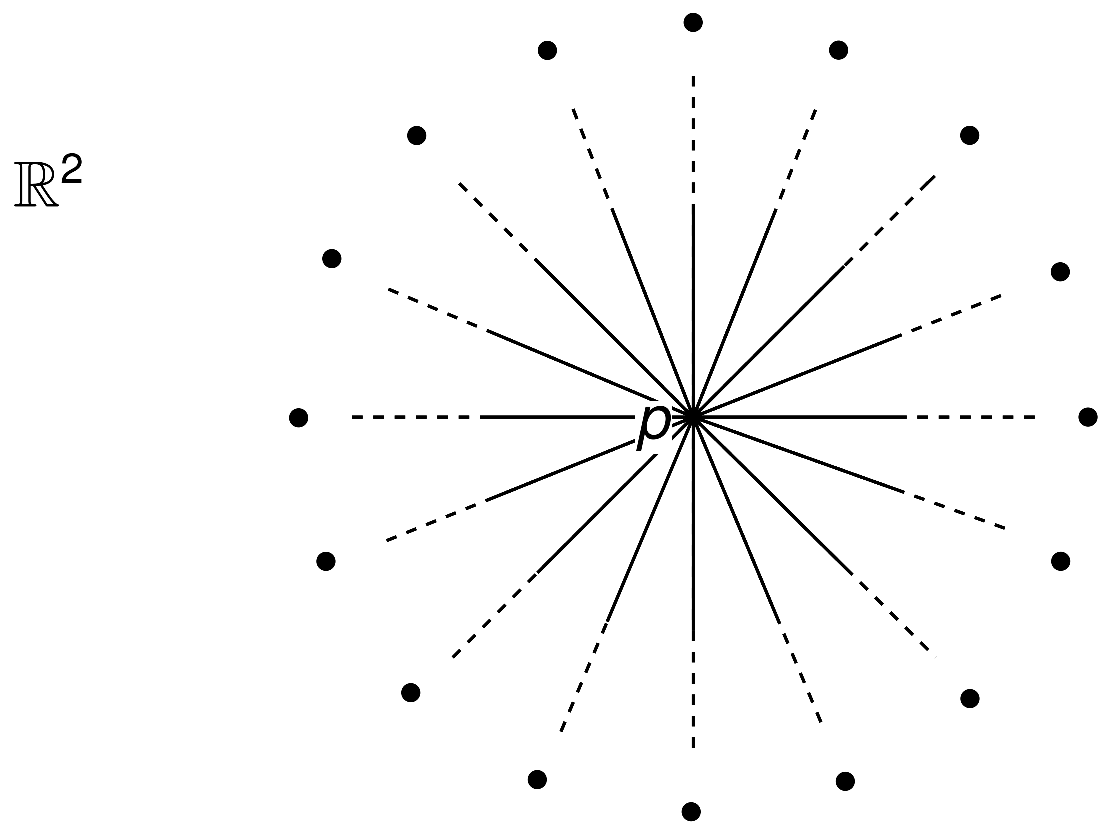

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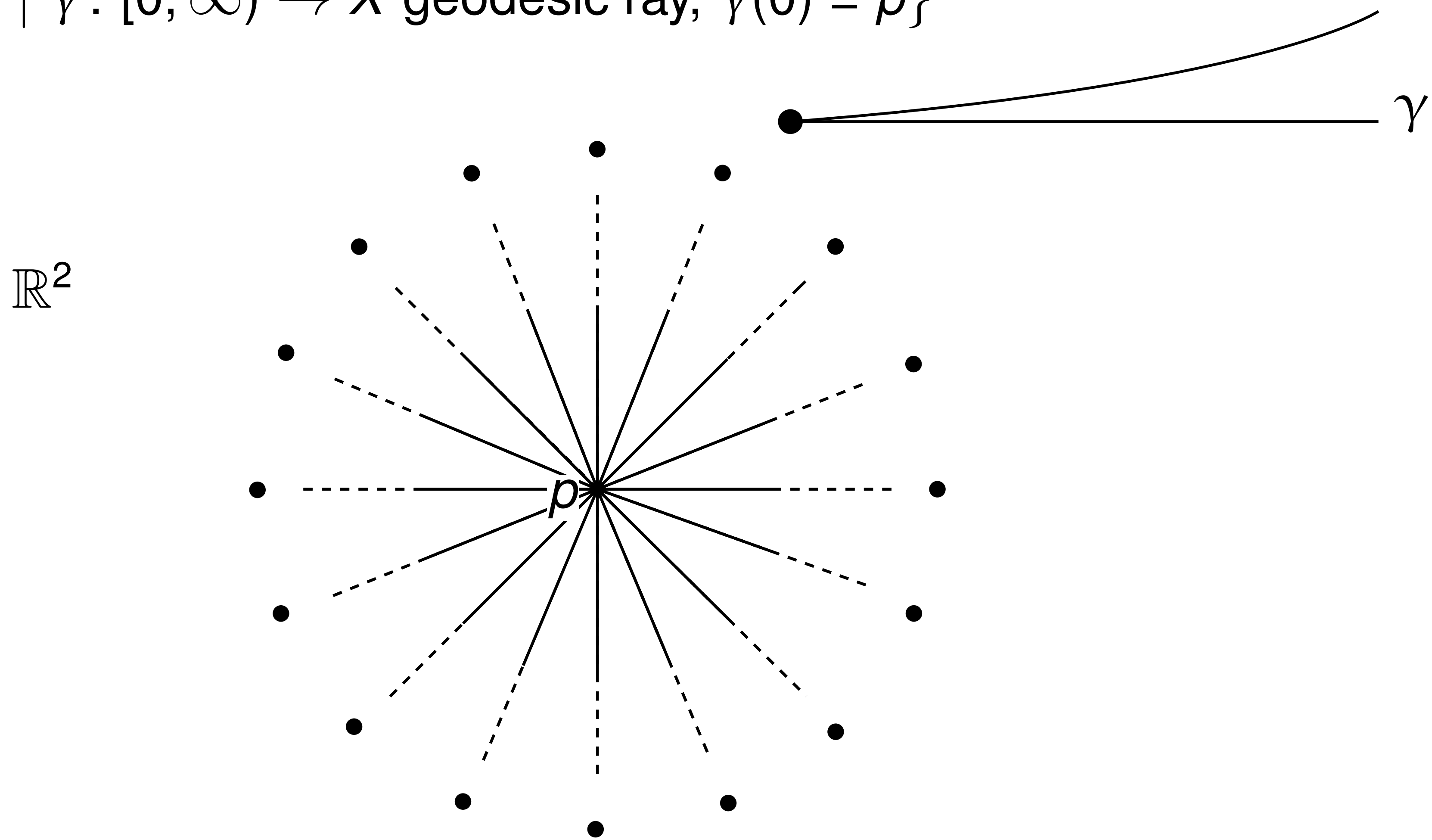

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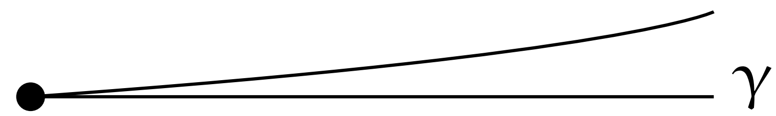

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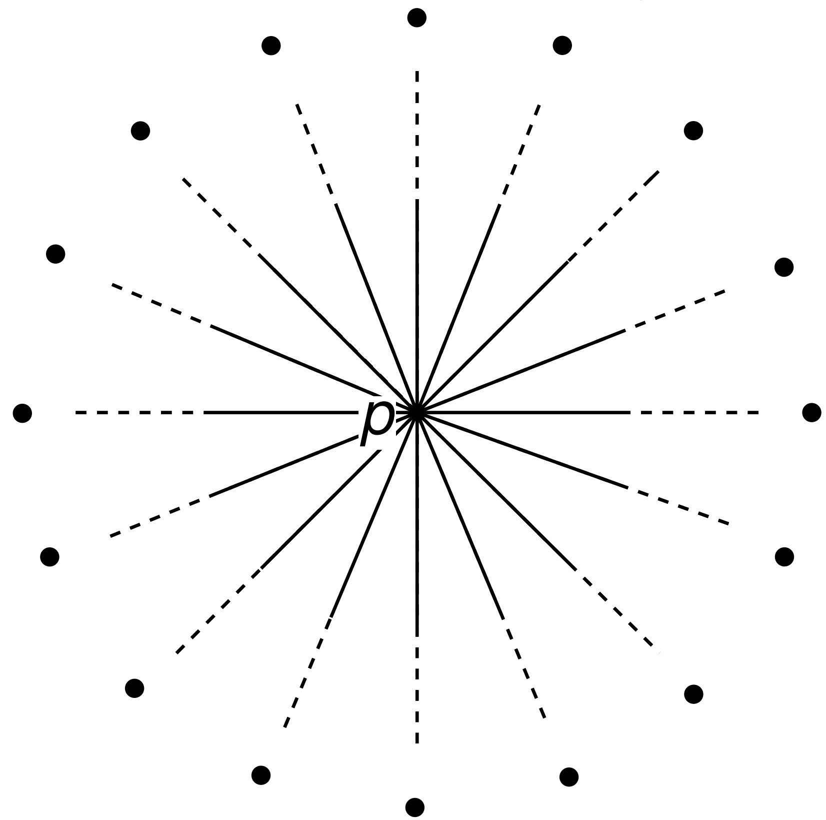

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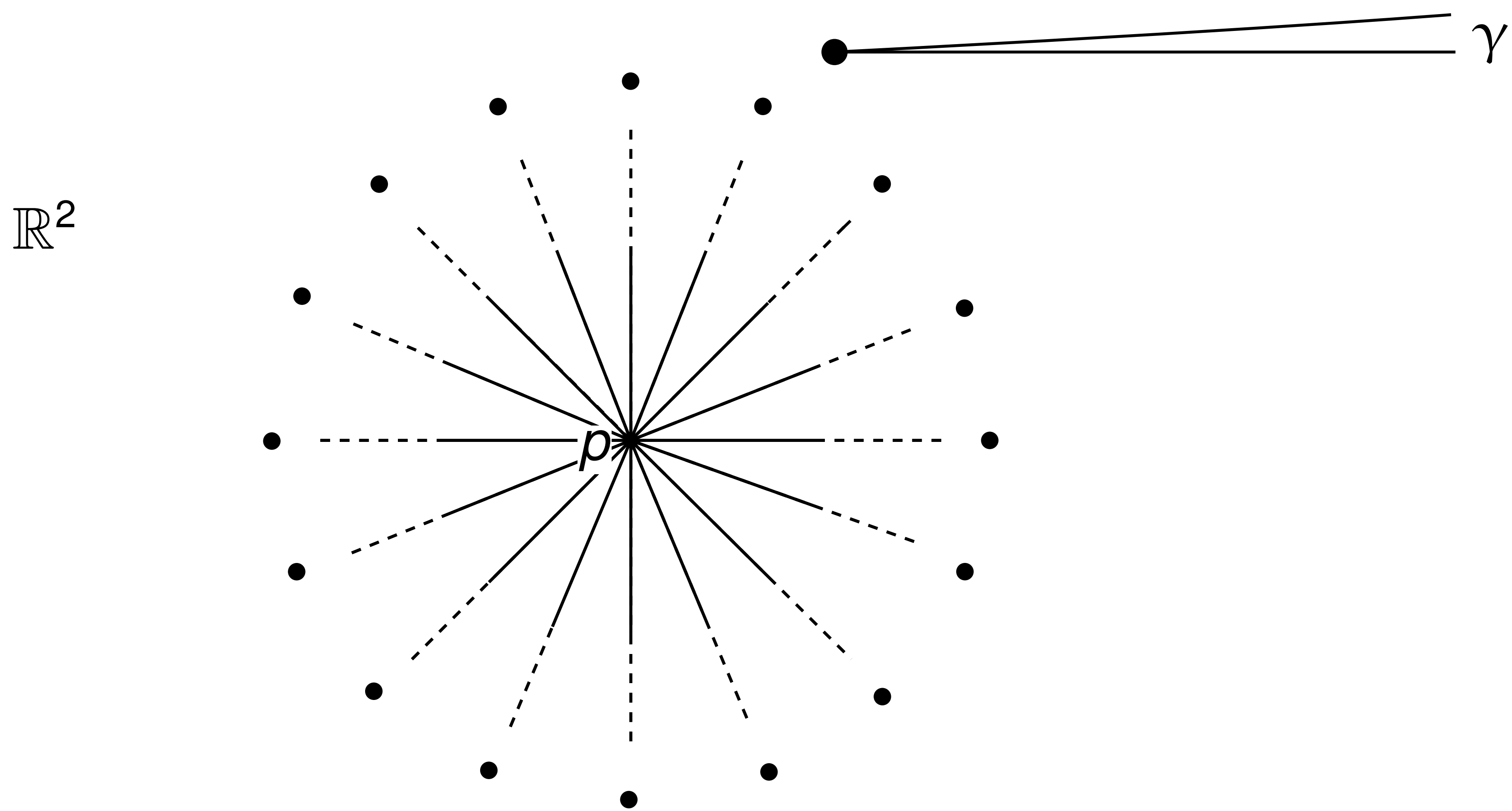

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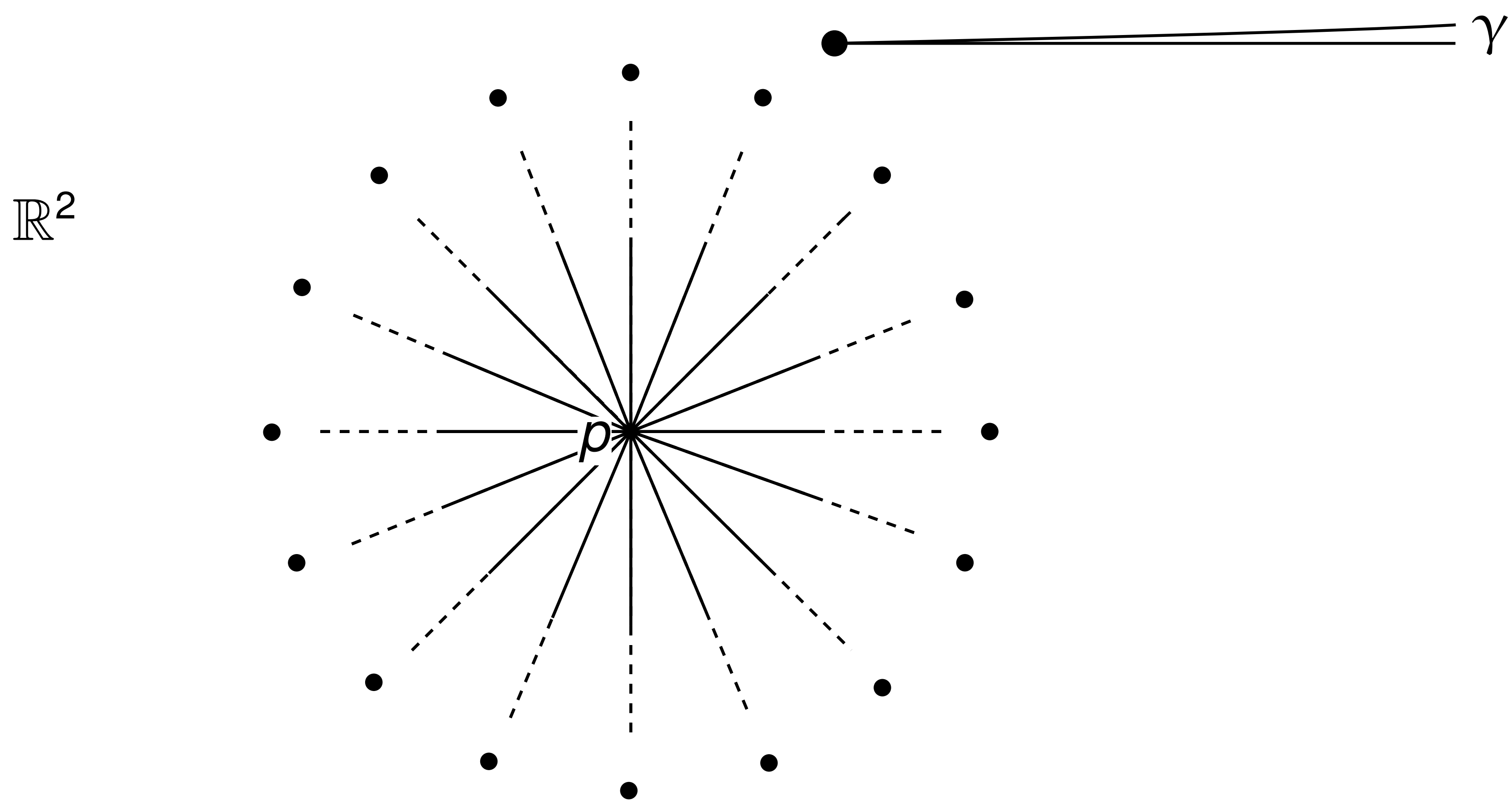

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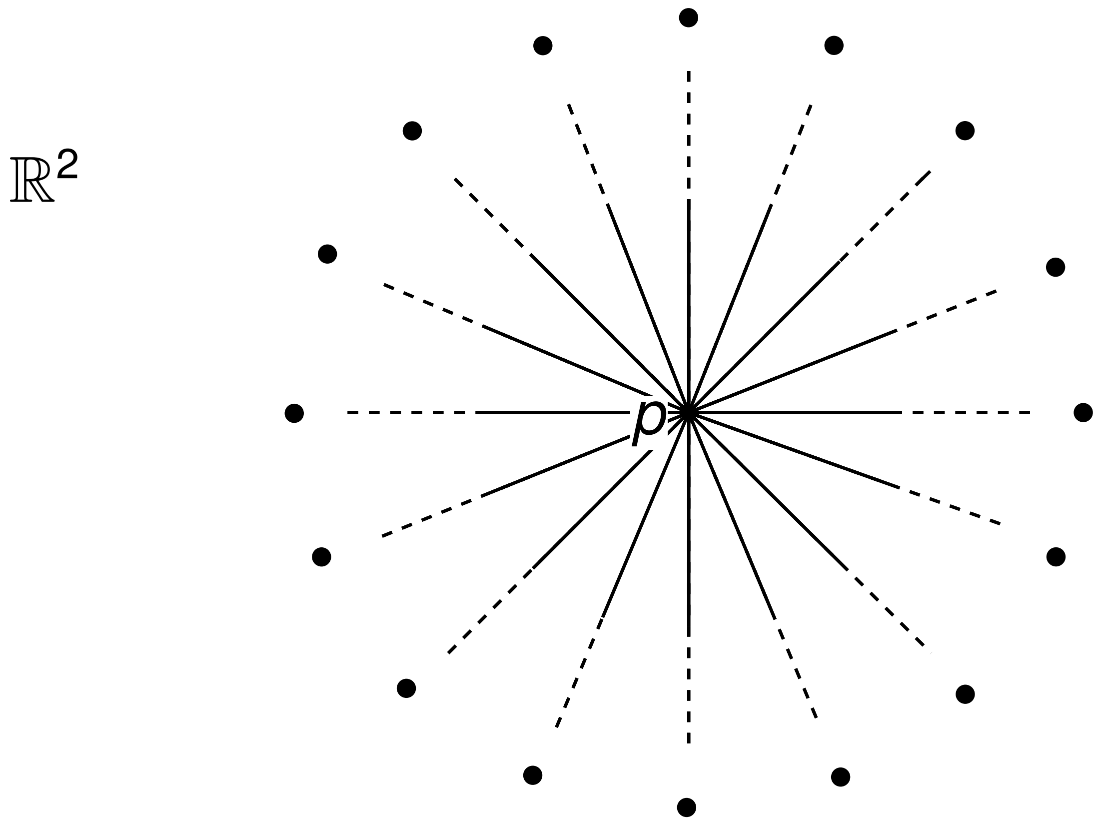

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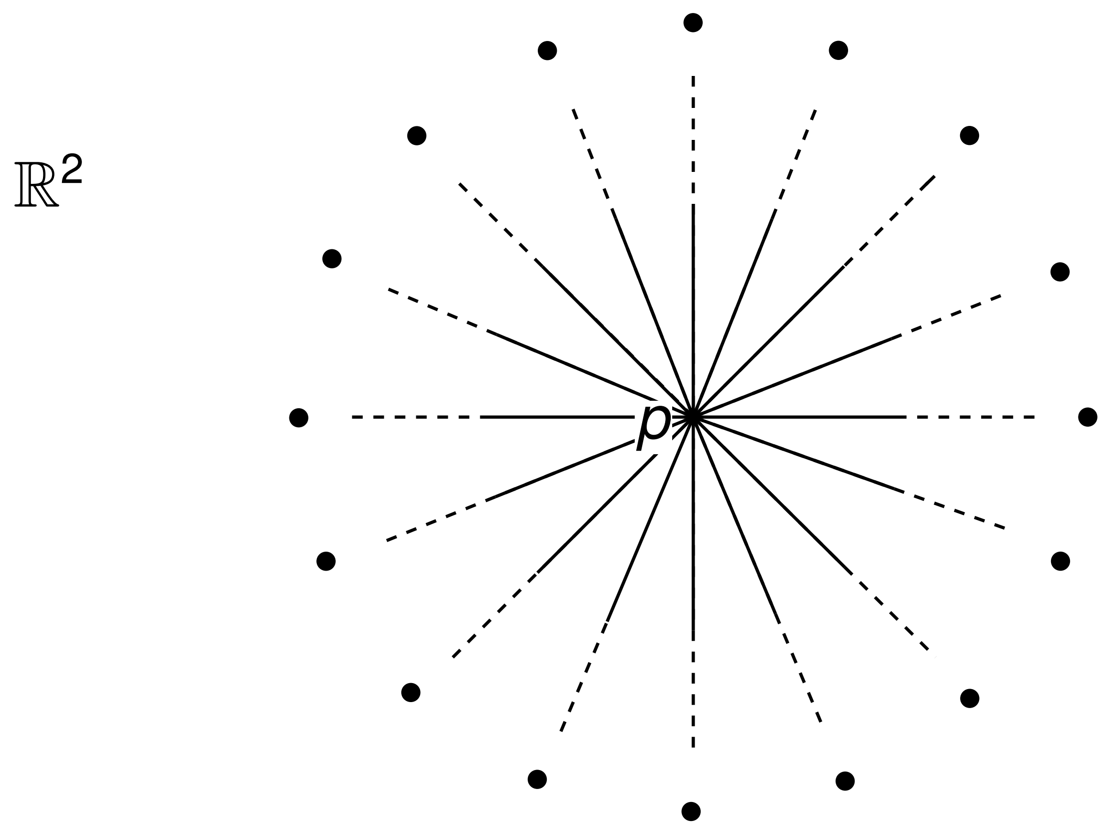

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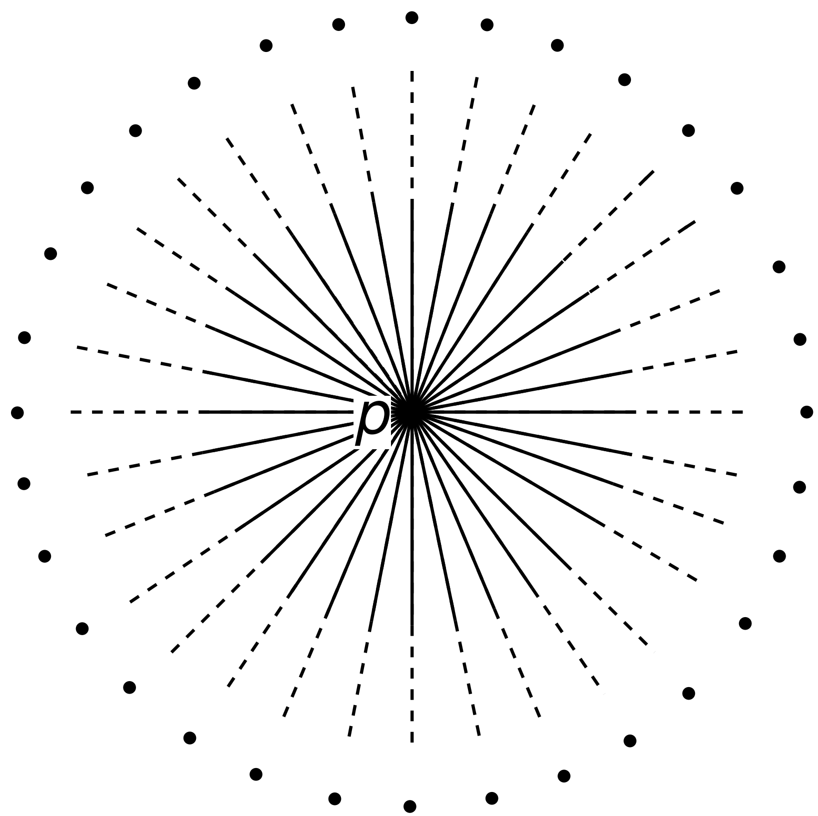

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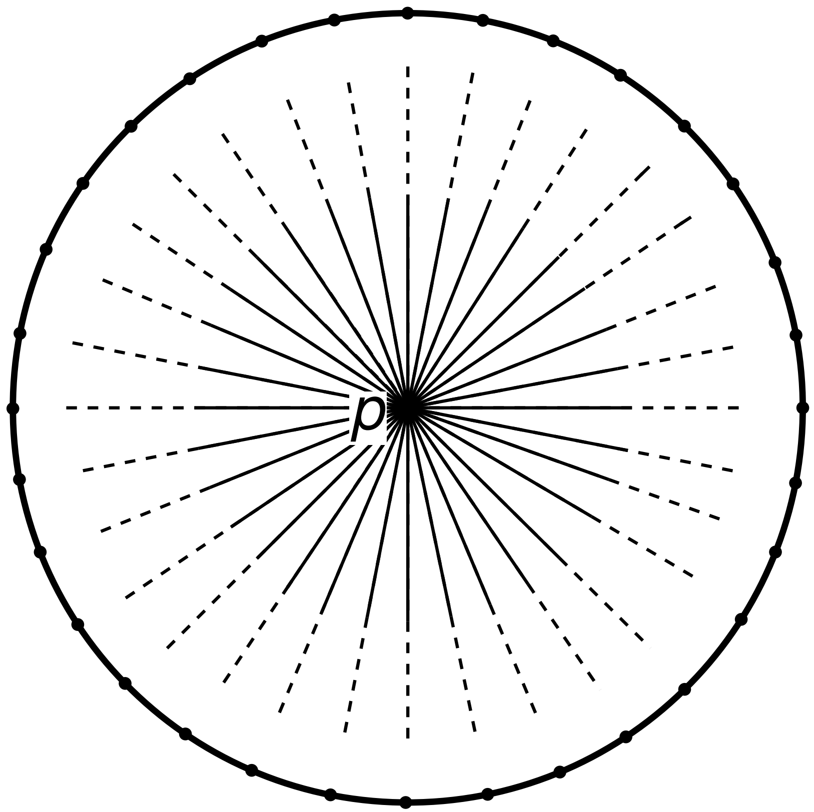

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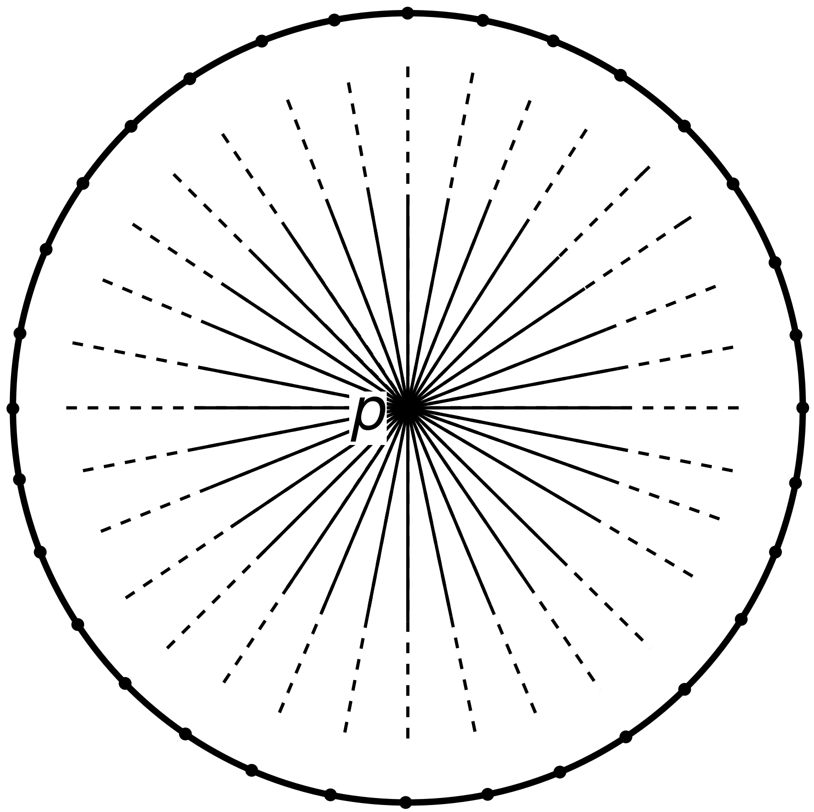

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Examples:

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Examples:

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Examples:

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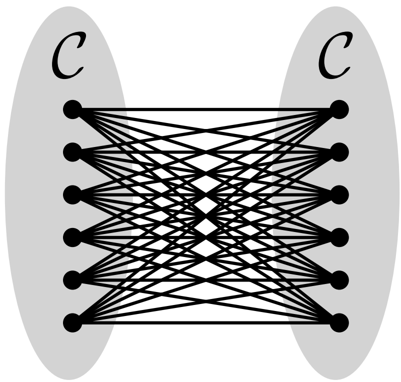
- $\partial \mathbb{H}^2 = S^1$,
- $\partial T_{n_{\geq 3}} = \mathcal{C}$ Cantor space,
- $X = X_1 \times X_2 \Rightarrow \partial X = \partial X_1 * \partial X_2$
- $X = T_{n_{\geq 3}} \times T_{m_{\geq 3}} \Rightarrow \partial X = \mathcal{C} * \mathcal{C}$



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CAT(0) group

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Gromov boundary of hyperbolic groups

X hyperbolic space, $\partial_{\mathrm{Grom}} X$ is defined similarly

X CAT(0) and hyperbolic $\Rightarrow \partial_{\mathrm{Grom}} X = \partial X$.

Theorem (Gromov):

$$\begin{tikzcd}[row sep=small] & \text{X hyperbolic} \\ \text{If G} \arrow[ur, bend left, "\text{acts geom.}"] \arrow[dr, bend right, "\text{acts geom.}"'] & \\ & \text{Y hyperbolic} \end{tikzcd} \qquad \text{then } \partial_{\mathrm{Grom}} X = \partial_{\mathrm{Grom}} Y.$$

Corollary: If G is hyperbolic, then $\partial G \coloneqq \partial X$ is well defined.

some space on which G acts geometrically

Visual boundary versus Gromov boundary

Theorem (Gromov):

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G has well-defined Gromov boundary

Theorem (Croke, Kleiner 00): $\exists G$ and CAT(0) spaces X, Y so that

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G does not have well-defined visual boundary

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Question: For which groups is the visual boundary well-defined?

Boundary rigidity

Definition (boundary rigidity): A CAT(0) group G is *boundary rigid* if

$$\begin{tikzcd} & X\ \mathrm{CAT}(0) \\ \text{If } G \arrow[ur, bend left, "\text{acts geom.}"] \arrow[dr, bend right, "\text{acts geom.}"'] & \\ & Y\ \mathrm{CAT}(0) \end{tikzcd} \quad \text{implies that } \partial X = \partial Y. \quad \rightsquigarrow \quad \begin{array}{l} G \text{ has well-defined} \\ \text{visual boundary} \end{array}$$

Examples of boundary rigid groups:

- $G = H \times \mathbb{Z}^n$ where H is hyperbolic (Bowers, Ruane 96)
- $G = H_1 \times H_2$ where H_1, H_2 are hyperbolic (Ruane 99)
- $G = H_1 \times H_2$ where H_1, H_2 are boundary rigid (Hosaka 2003)
- G acts geom. on a CAT(0) space that is relatively hyperbolic with respect to a family of flats. (Hruska–Kleiner 2005)

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Goal: Lattices in products of trees are boundary rigid!

Lattices in products of trees

Definition: A group G is a *lattice* in $T_n \times T_m$ if G acts geom. on $T_n \times T_m$.

regular trees,
deg≥ 3

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Theorem: There exist irreducible lattices!
(They were first studied by Burger, Mozes, Wise)

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Theorem: There exist irreducible lattices!

Burger, Mozes, Wise

Burger–Mozes 97

first example of a simple CAT(0) group

Lattices in products of trees are boundary rigid

Main Theorem: $T_n \times T_m$ has a single orbit of vertices.

If G acts vertex-transitively and freely on $T_n \times T_m$ then G is boundary rigid.

regular trees,
deg≥ 3

In particular, $\partial G = \mathcal{C} * \mathcal{C}$.

Lattices in products of trees are boundary rigid

Main Theorem:

$T_n \times T_m$ has a single orbit of vertices.

If G acts vertex-transitively and freely on $T_n \times T_m$ then G is boundary rigid.

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In particular, $\partial G = \mathcal{C} * \mathcal{C}$.

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Starting point:

$$\begin{tikzcd}[column sep=small]
 & G \arrow[dl, bend right, "\text{vertex-transitively}\\\text{freely}"'] \arrow[dr, bend left, "\text{geometrically}"] & \\
T_n \times T_m & & X \text{ CAT(0)}
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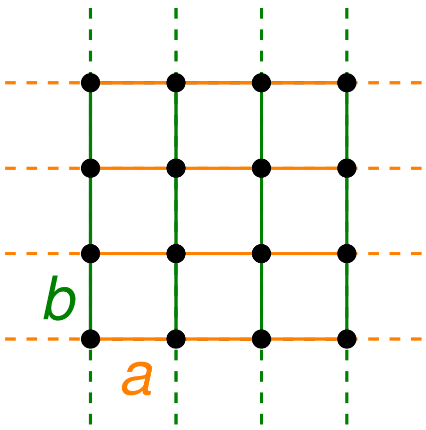
$$\begin{array}{ccc} & G & \\ \text{vertex-transitively, freely} \swarrow & & \searrow \text{geometrically} \\ T_n \times T_m & & X \text{ CAT}(0) \end{array}$$

- $dim(\partial(T_n \times T_m)) = 1 \overset{\text{Geoghegan, Ontaneda}}{\Rightarrow} dim(\partial X) = 1$
- vertex-transitively freely $\overset{\text{Wise}}{\Rightarrow}$ $\exists a, b \in G$ s.t. $\langle a, b \rangle \cong \mathbb{Z}^2 \curvearrowright F \underset{\text{convex}}{\subseteq} X$ flat



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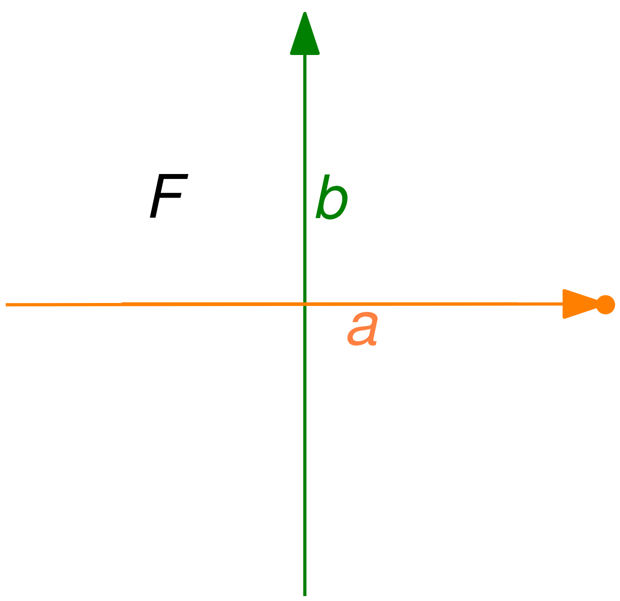

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Strategy: Use Dynamics on Tits boundaries (Guralnik, Swenson, Ricks)

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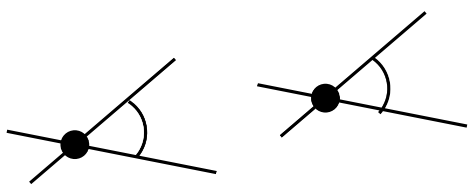
Definition (Tits boundary): Let X CAT(0).

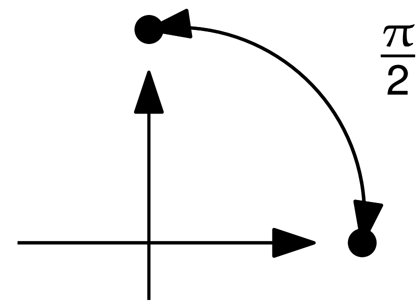
$\partial_T(X) \coloneqq (\partial X, d_T)$ is the *Tits boundary* of X.

Tits distance

$\mathbb{R}^2$



$\partial_T \mathbb{R}^2 = S^1$



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Strategy: Use Dynamics on Tits boundaries (Guralnik, Swenson, Ricks)

space X	∂X	$\partial_T X$
$\mathbb{R}^2$	○	◇
$\mathbb{H}^2$	○	discrete
tree T	$\mathcal{C}$	discrete

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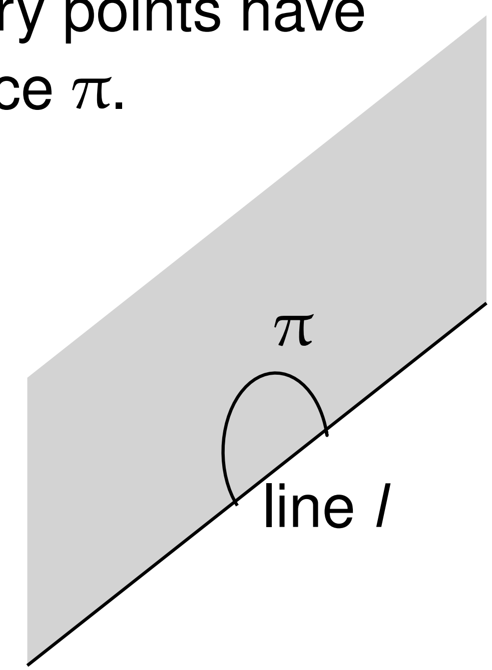
a line bounds a half flat iff its boundary points have Tits distance π.



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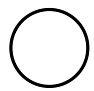
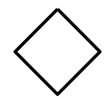
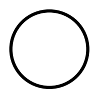
■ $\mathbb{R}^2$		
■ $\mathbb{H}^2$		discrete
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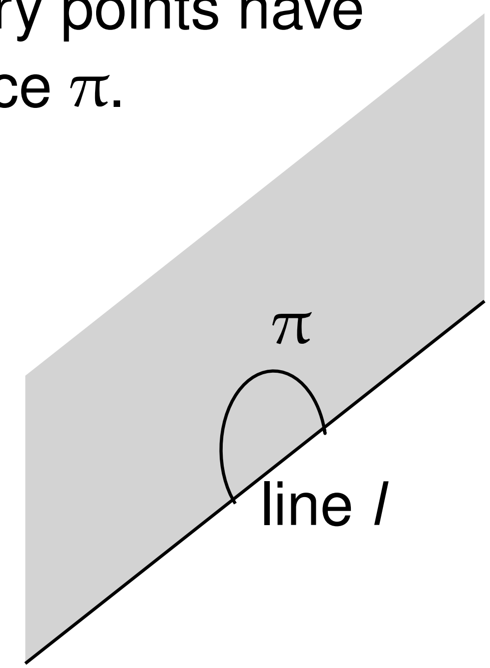

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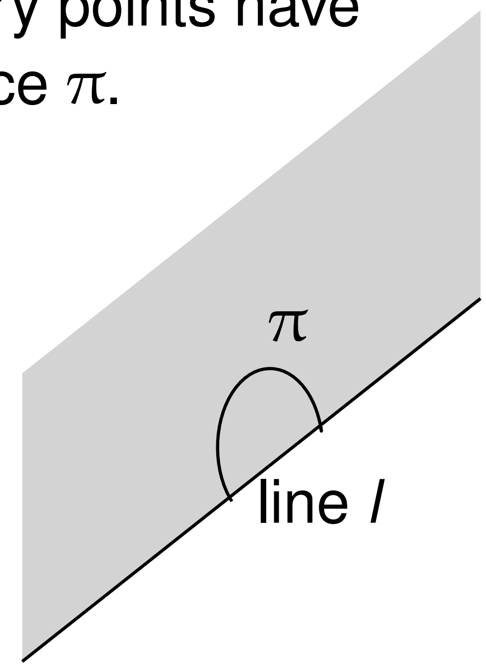

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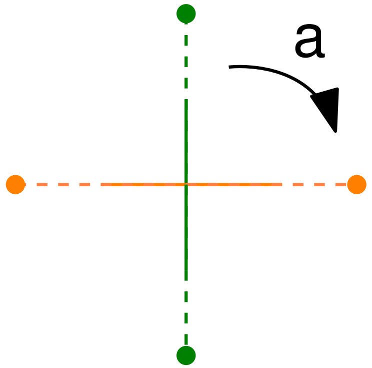

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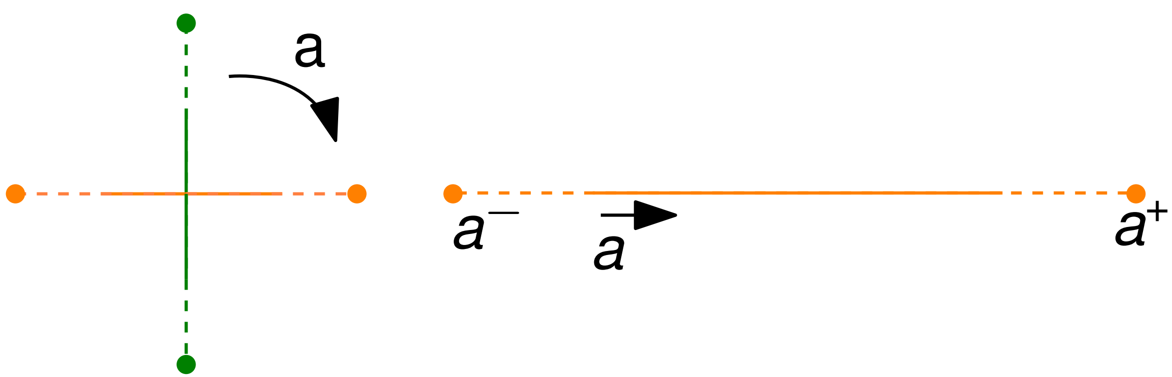

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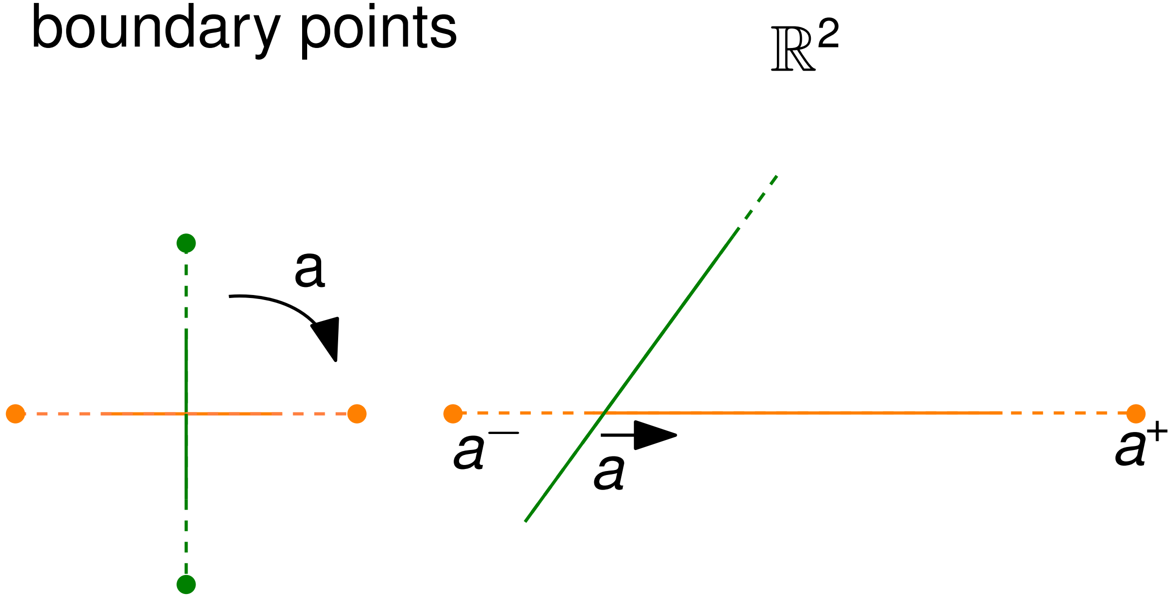

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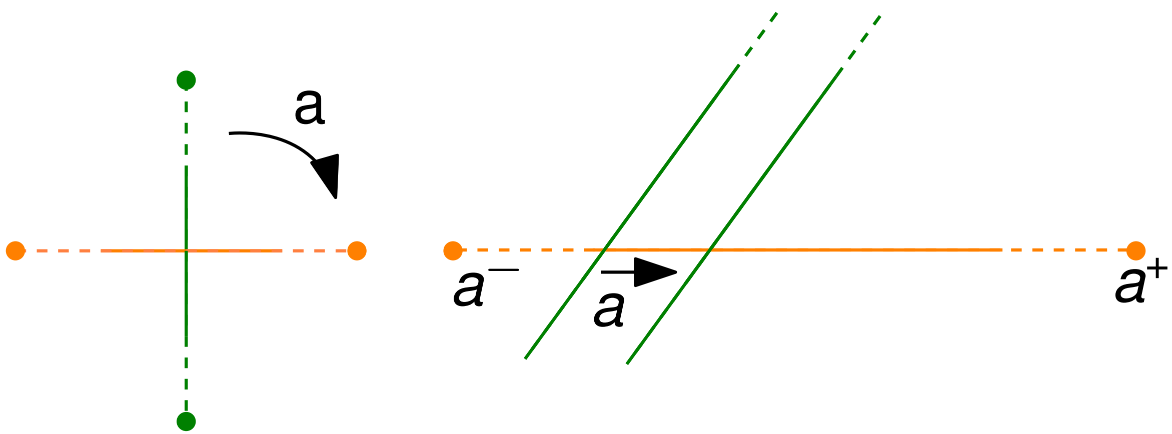

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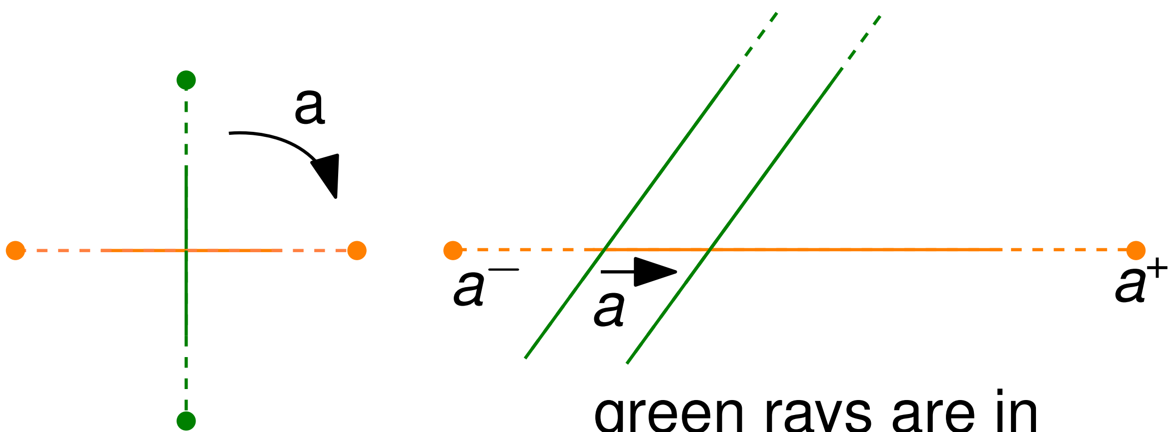

green rays are in
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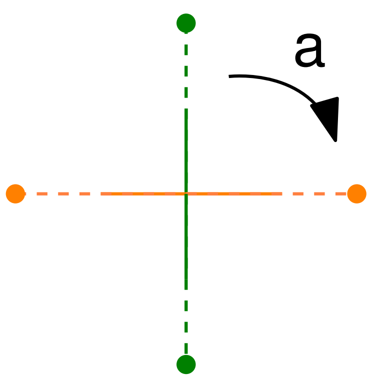

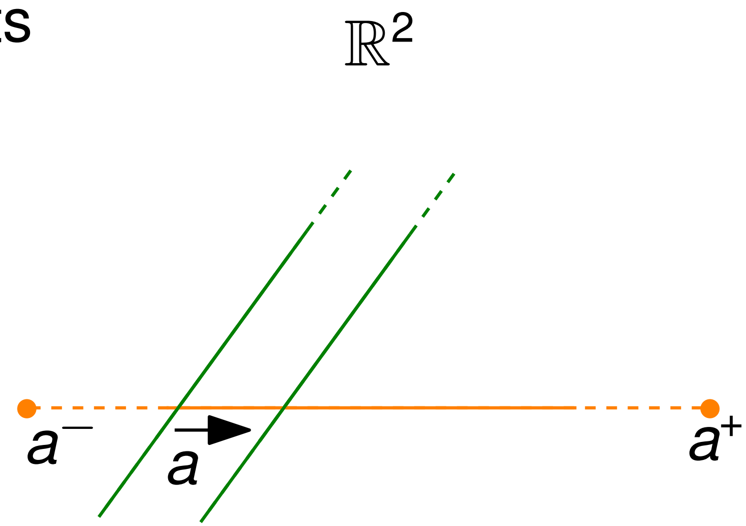

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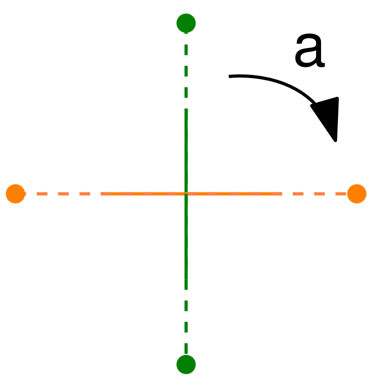

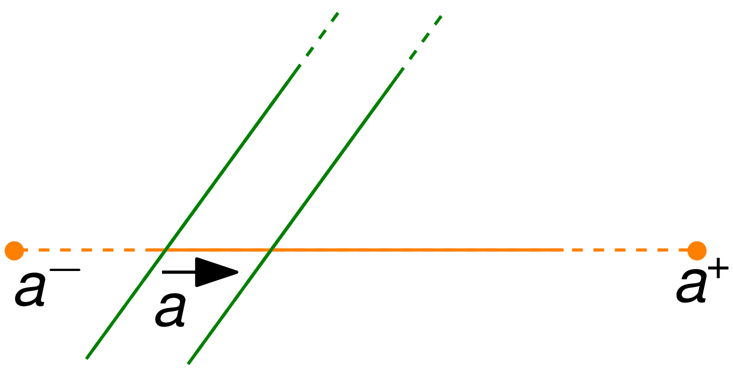

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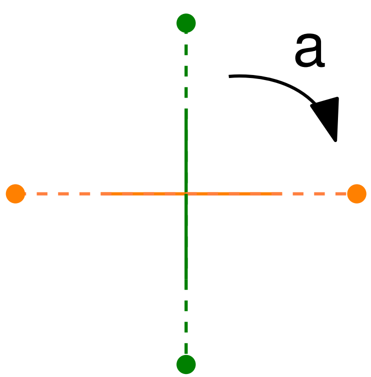

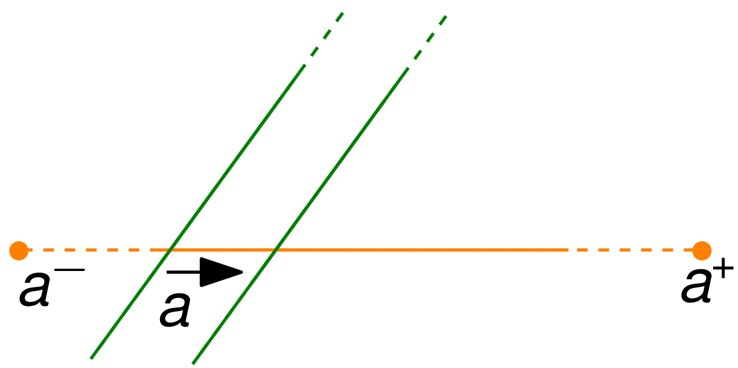

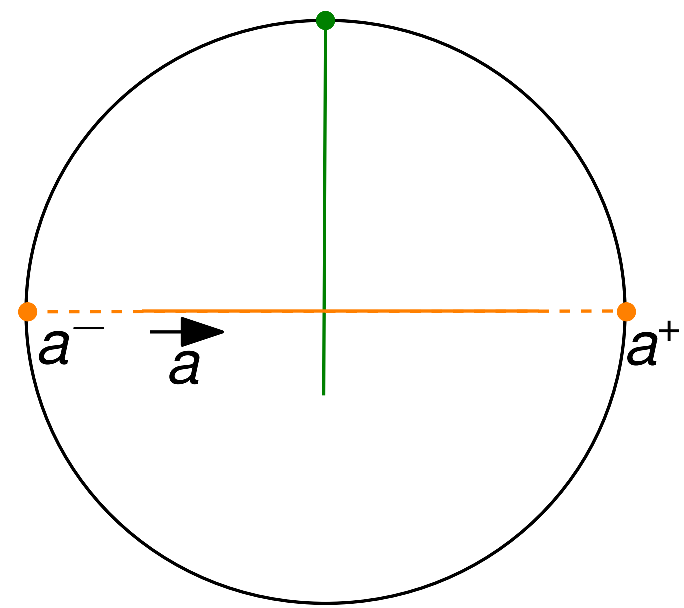

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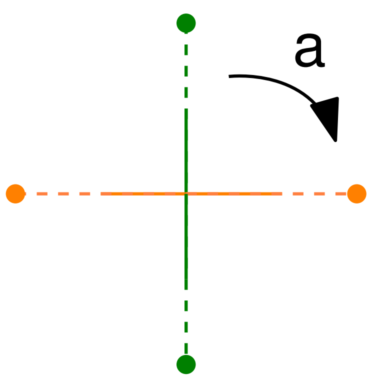

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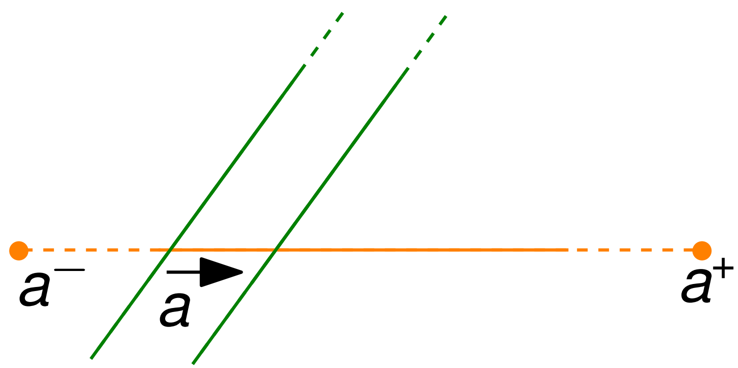

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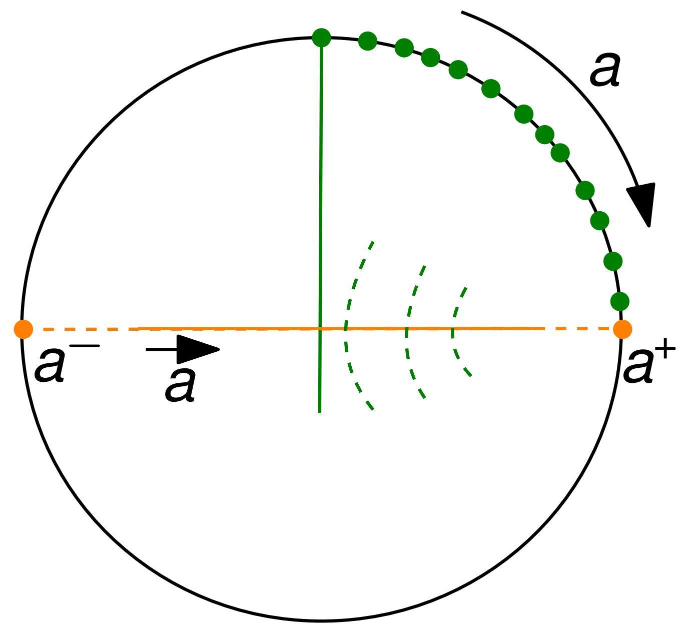
North-South-dynamics

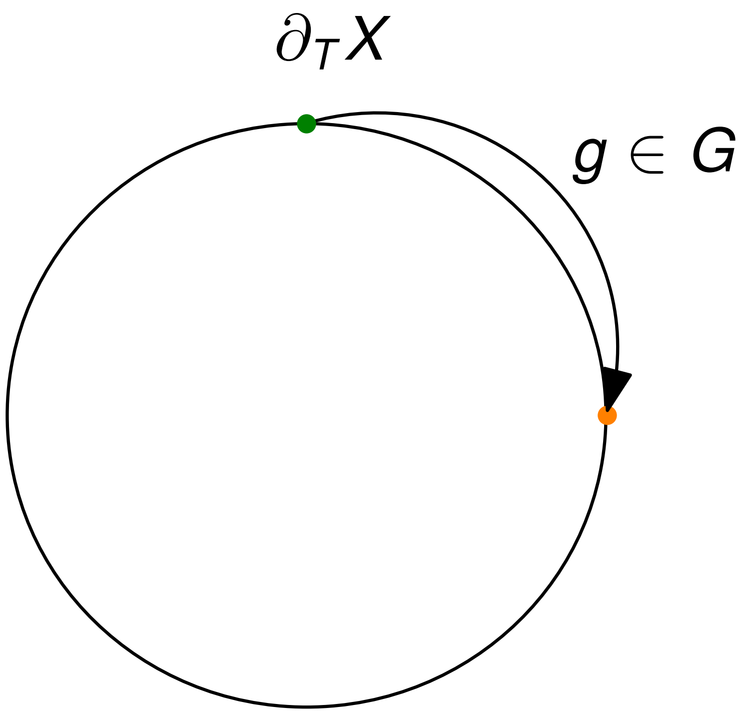
$\mathbb{H}^2$



$a^n a^- \xrightarrow{n\to\infty} a^-$,
$a^n\bar{\gamma} \xrightarrow{n\to\infty} a^+$
for all $\bar{\gamma} \in \partial X - \{a^-\}$

Proof sketch of Claim 1

Generalize dynamics on boundaries (Guralnik–Swenson, 2013):



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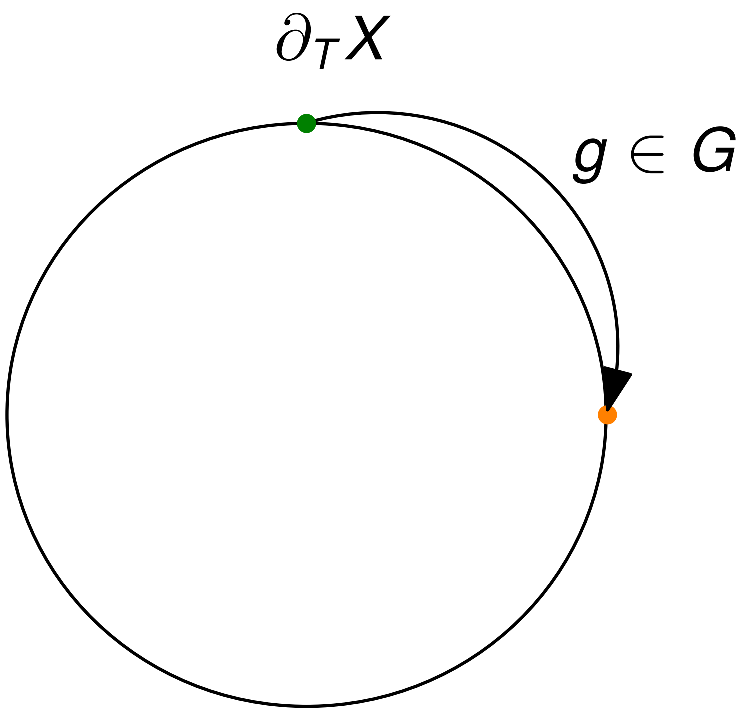
Generalize dynamics on boundaries (Guralnik–Swenson, 2013):

$\beta G :=$ Stone-Čech compactification
$= $ set of ultrafilters on G



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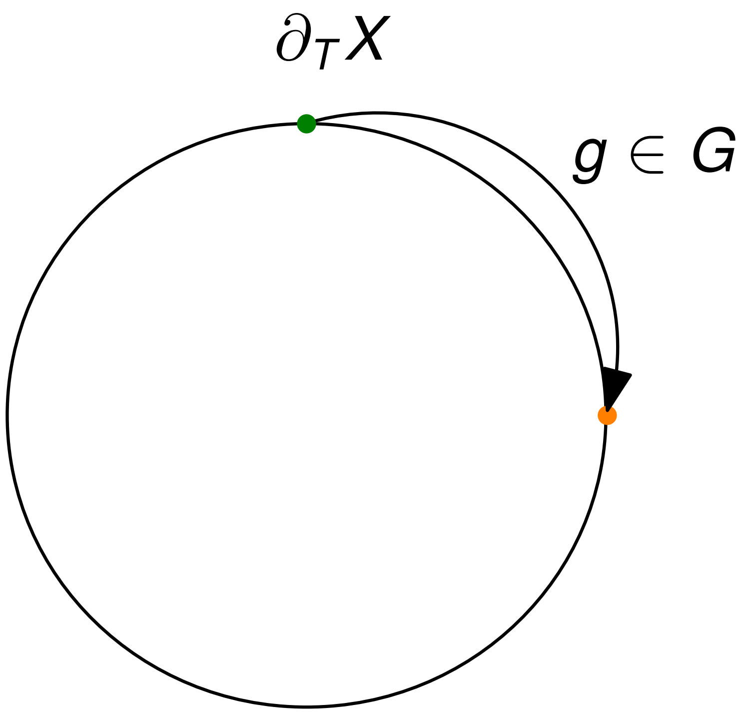

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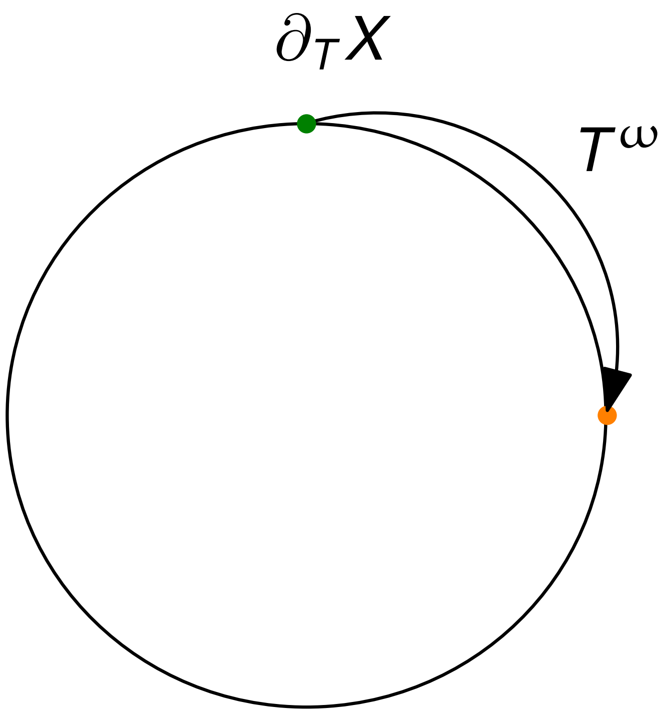
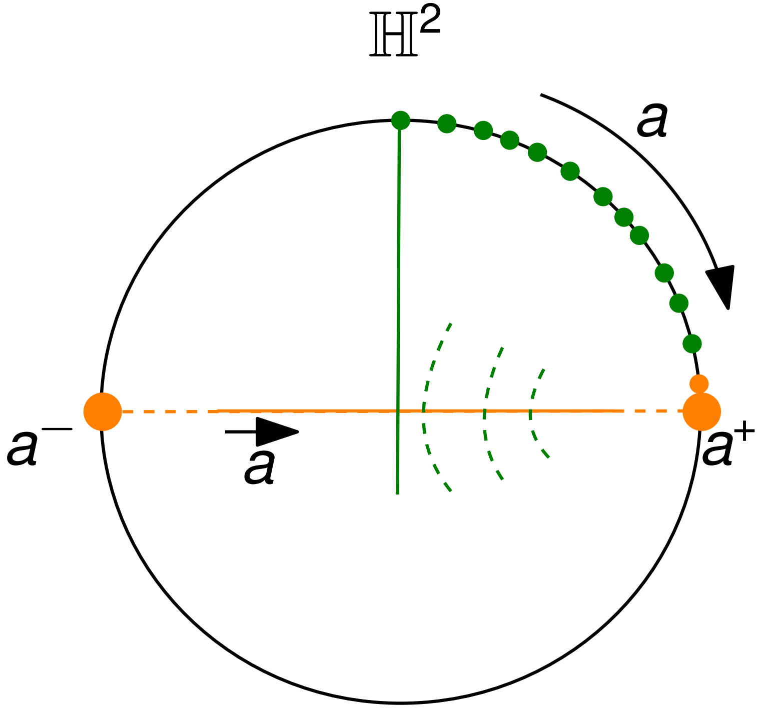
$\omega \mapsto T^{\omega} : \partial_T X \to \partial_T X$



Proof sketch of Claim 1

Theorem (a higher-dimensional version of North-South dynamics):



North-South-dynamics

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Theorem (a higher-dimensional version of North-South dynamics):
Suppose that

- G acts geometrically on a CAT(0) space X,
- $d \coloneqq dim(\partial_T X)$,
- $\exists H < G$, $H \cong \mathbb{Z}^{d+1}$.

Then $\exists$ $(d+1)$-flat $F \underset{\text{convex}}{\subseteq} X$ and $\omega \in \beta G$ so that

- F is H-invariant,
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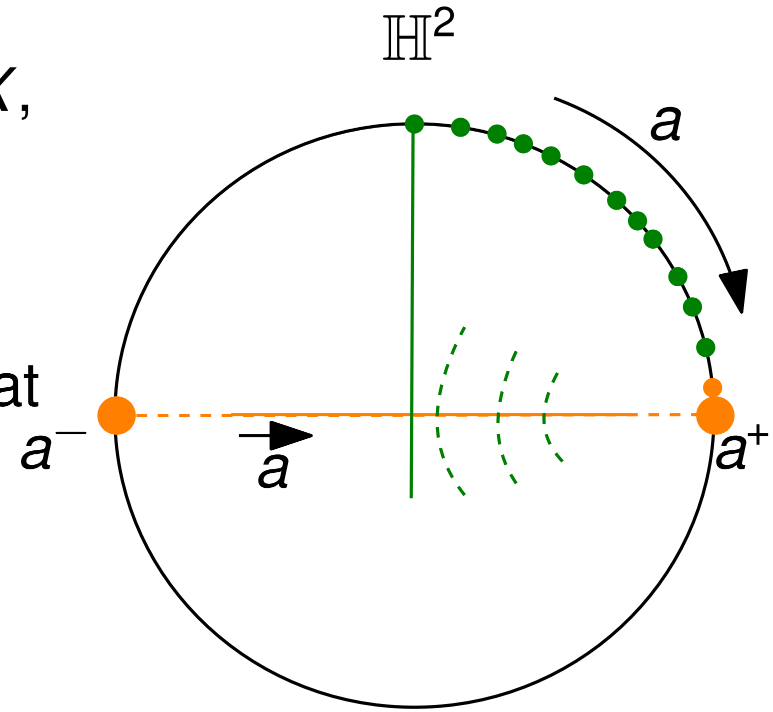

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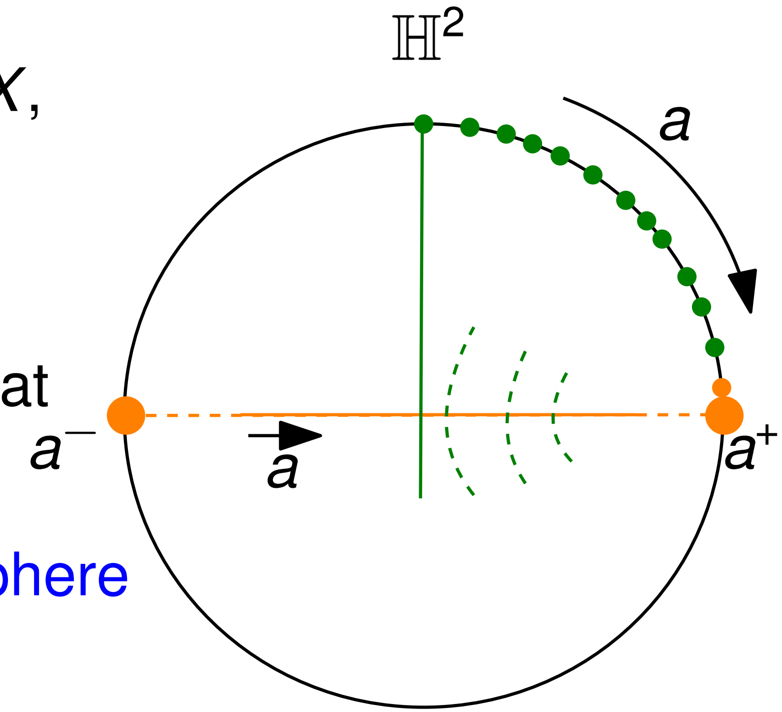

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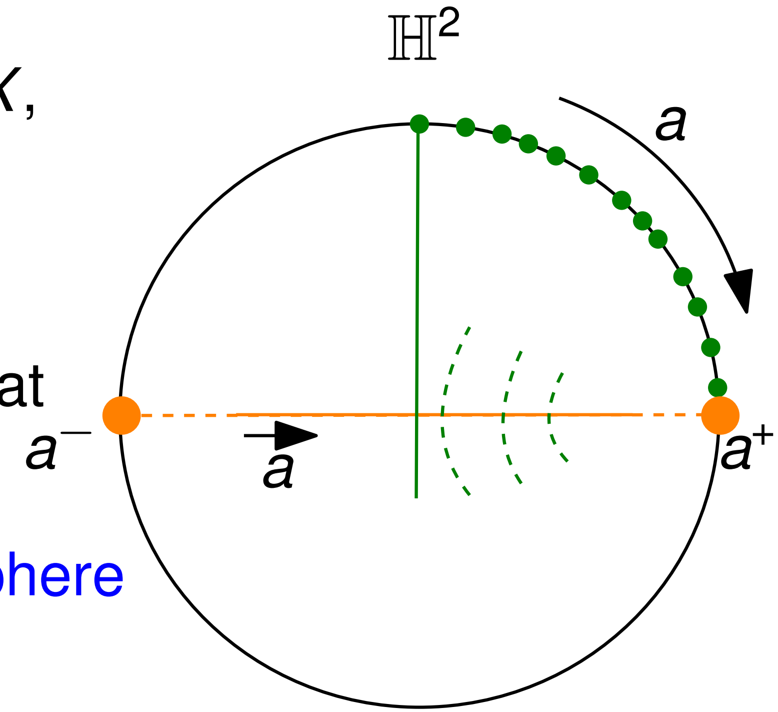

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X proper hyperbolic $\Rightarrow \partial_T X$ is 0-dimensional

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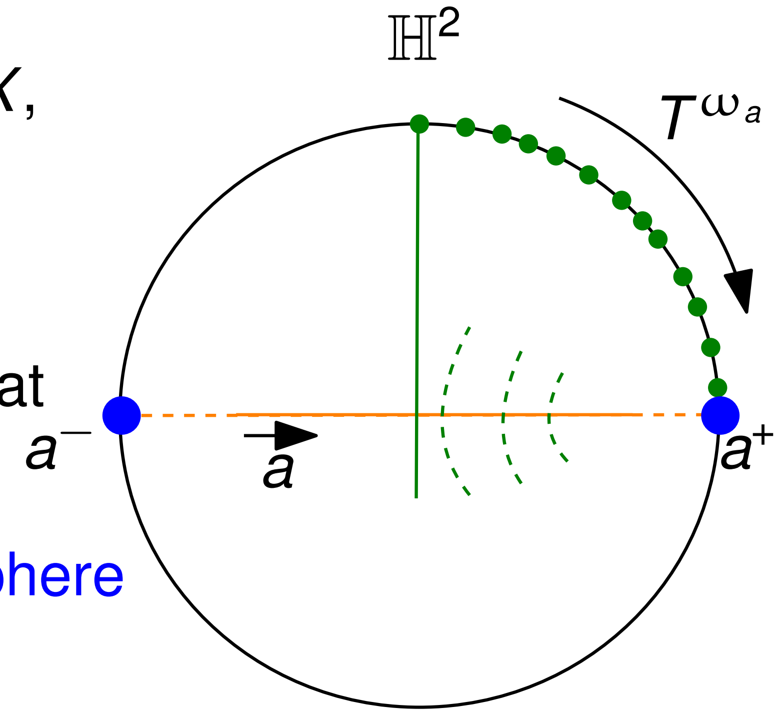

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X proper hyperbolic $\Rightarrow$ $\partial_T X$ is 0-dimensional
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Then $\exists$ $(d+1)$-flat $F \subseteq X$ and $\omega \in \beta G$ so that
convex

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- $T^{\omega}|_{\partial_T F} = \mathrm{id}$.

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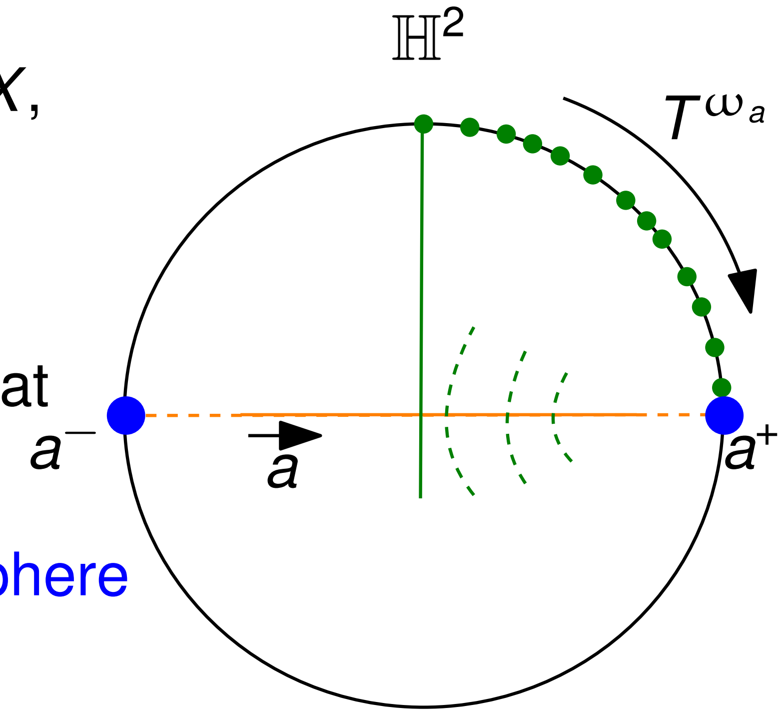

North-South-dynamics

Proof sketch of Claim 1

Theorem (a higher-dimensional version of North-South dynamics):
Suppose that

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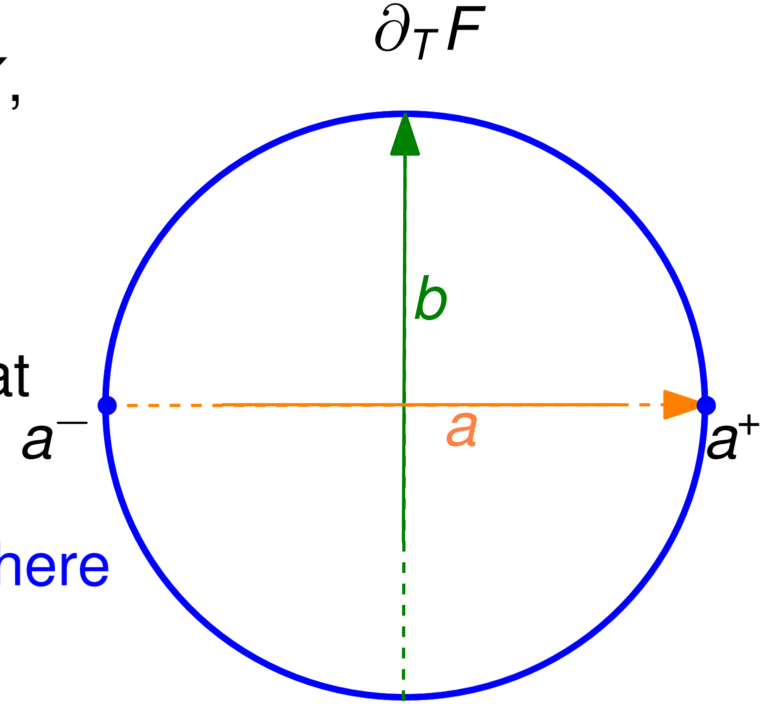

$d = 1$: $\exists a, b \in G$ s.t.

$\langle a, b \rangle \cong \mathbb{Z}^2 \curvearrowright F \subseteq X$ flat

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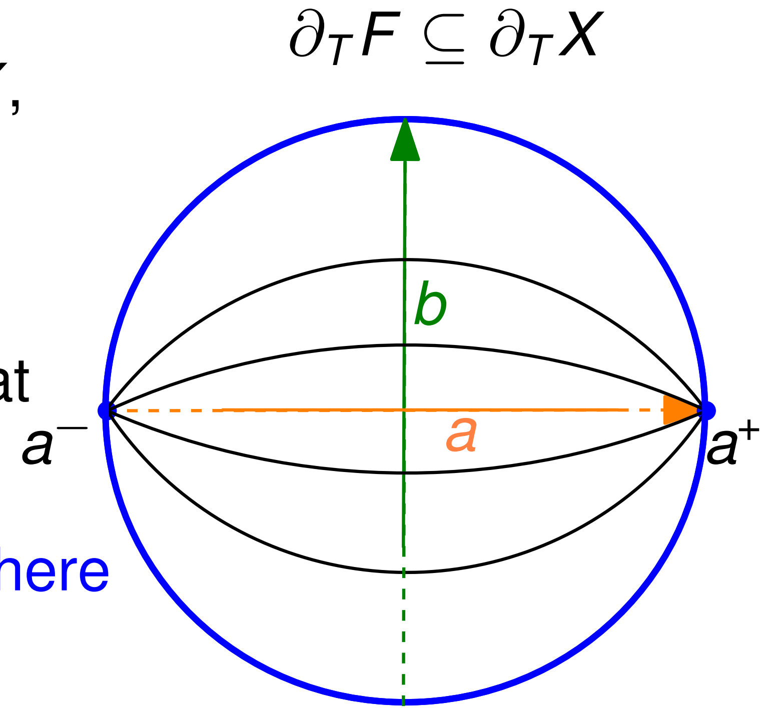

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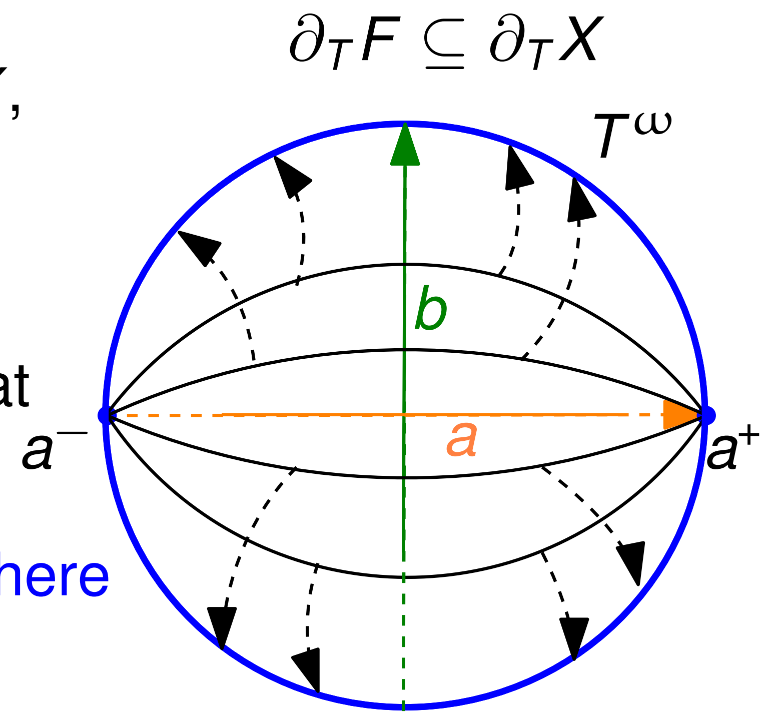
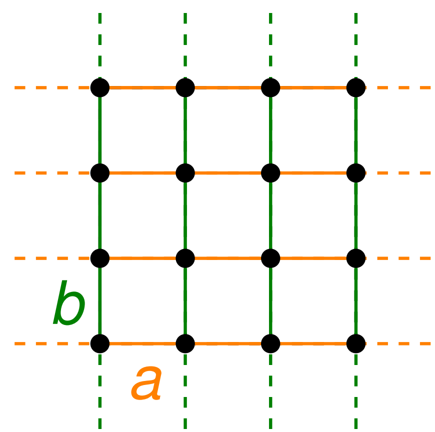


Proof sketch of Claim 1

Starting point:

$$\begin{tikzcd} & G \arrow[dl, "\text{vertex-transitively freely}"'] \arrow[dr, "\text{geometrically}"] & \\ T_n \times T_m & & X \text{ CAT}(0) \end{tikzcd}$$

- $dim(\partial(T_n \times T_m)) = 1$ $\overset{\text{Geoghegan, Ontaneda}}{\Rightarrow}$ $dim(\partial X) = 1$
- vertex-transitively freely $\overset{\text{Wise}}{\Rightarrow}$ $\exists a, b \in G$ s.t. $\langle a, b \rangle \cong \mathbb{Z}^2 \curvearrowright F \underset{\text{convex}}{\subseteq} X$ flat

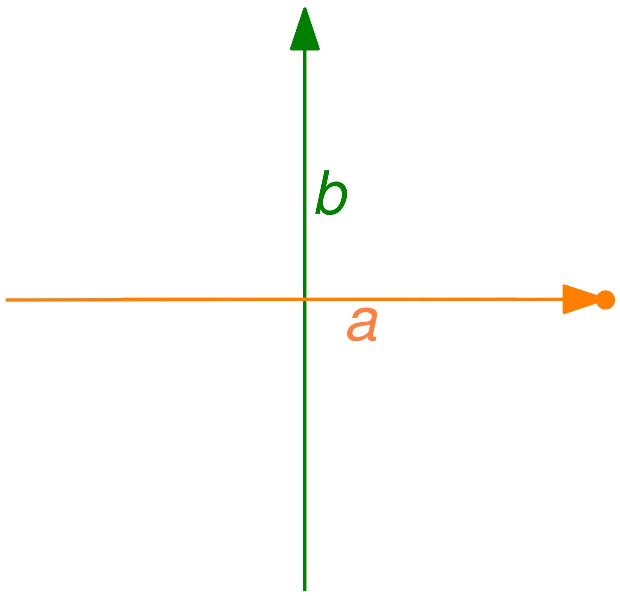


Proof sketch of Claim 1

Starting point:

$$G \overset{\text{geom.}}{\curvearrowright} X \text{ CAT(0)}$$

- $dim(\partial X) = 1$
- $\exists a, b \in G$ s.t. $\langle a, b \rangle \cong \mathbb{Z}^2 \curvearrowright \underbrace{F \subseteq X}_{\text{convex}}$ flat



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$G \overset{\text{geom.}}{\curvearrowright} X$ CAT(0)

- $dim(\partial X) = 1$
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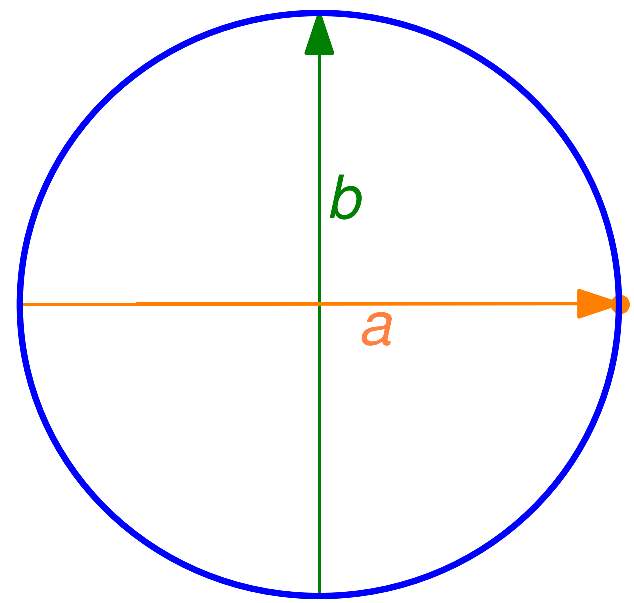
  convex

$\partial_T F \subseteq \partial_T X$



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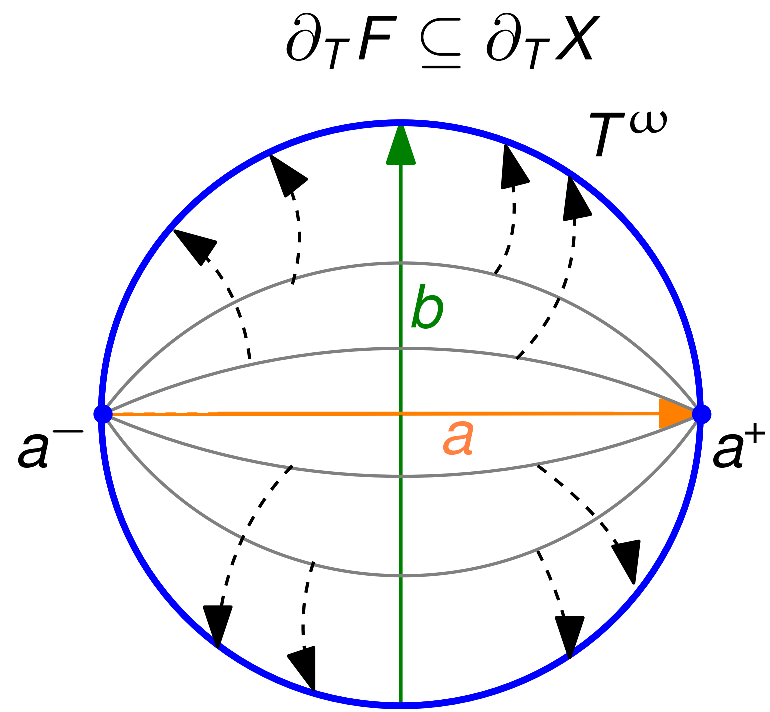

$\Rightarrow \partial_T F$ is an attracting top-dimensional sphere!

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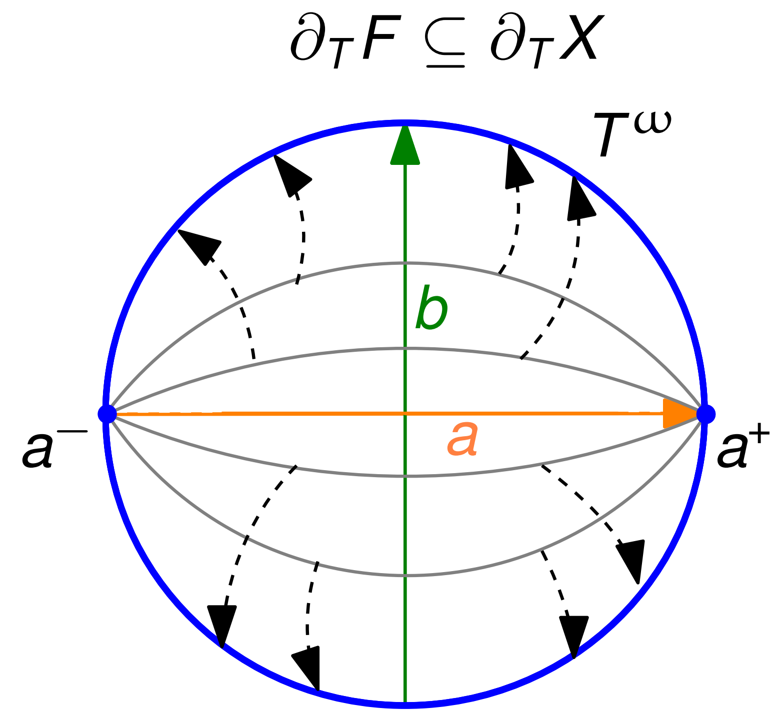
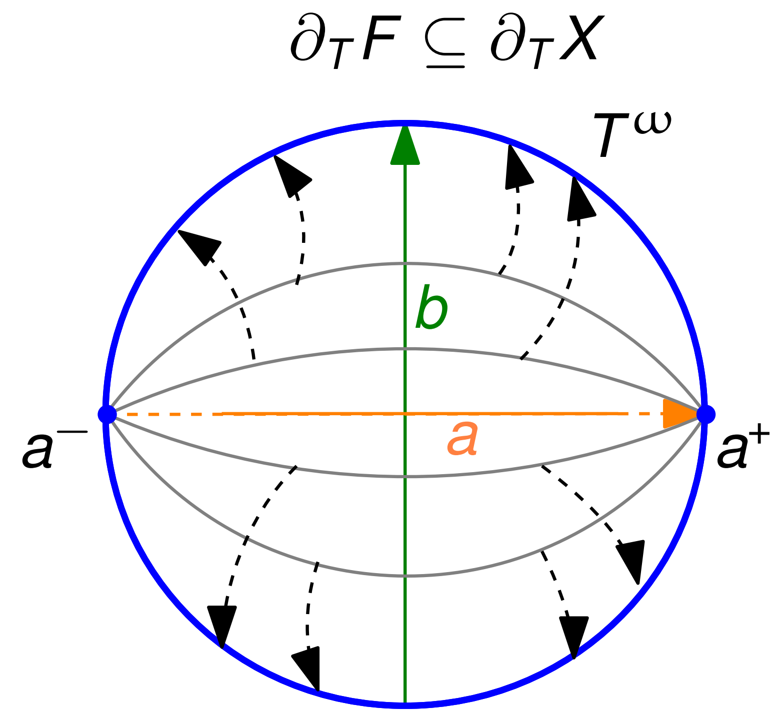


$\Rightarrow \partial_T F$ is an attracting top-dimensional sphere!

Now, apply a splitting criterion of Ricks (2020)!

Thank you for your attention!



Proof sketch of Claim 1

Strategy: Use Dynamics on Tits boundaries (Guralnik, Swenson, Ricks)

Definition (Tits boundary): Let X CAT(0).

$\partial_T(X) \coloneqq (\partial X, d_T)$ is the *Tits boundary* of X.

Tits distance:

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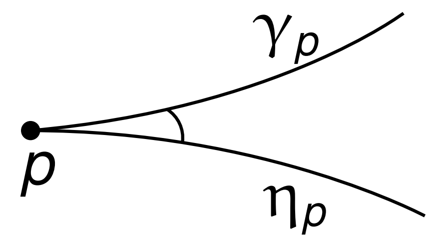
$\angle_p : \partial X \times \partial X \to [0, \pi]$

$\angle_p(\bar{\gamma}, \bar{\eta}) \coloneqq$ angle between representatives at p



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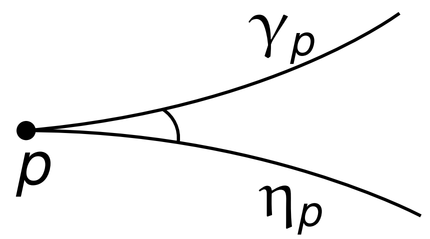

$\angle(\bar{\gamma}_1, \bar{\gamma}_2) = \sup_{x \in X} \angle_x(\bar{\gamma}, \bar{\eta})$

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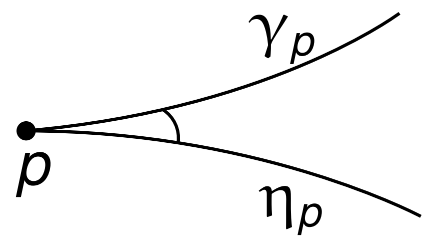

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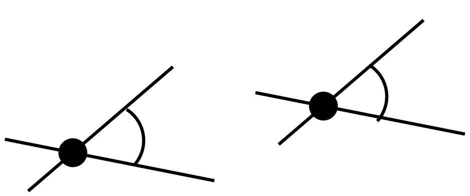
$\mathbb{R}^2$



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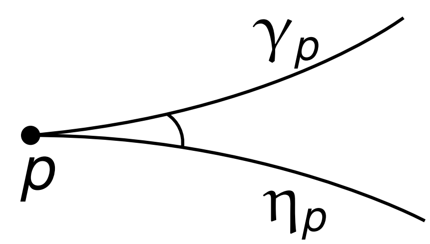

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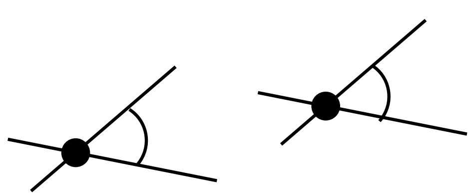

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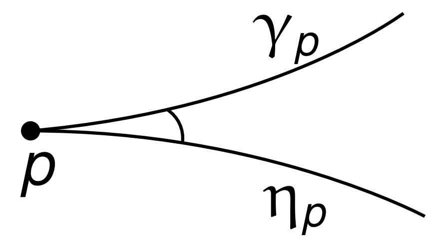

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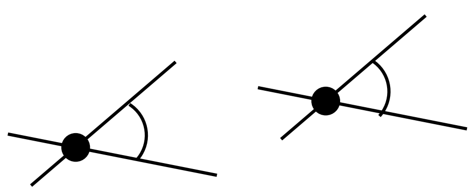
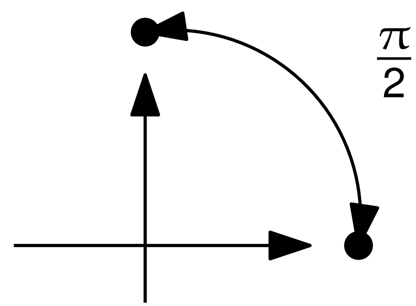


$\partial_T \mathbb{R}^2 = S^1$



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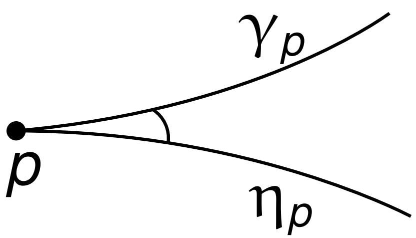

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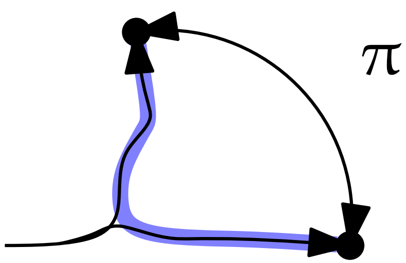
d_T = length metric iduced by $\angle$

$\partial_T(tree)$ = discrete space



Proof sketch of Claim 1

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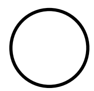
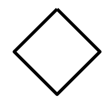
space X	∂X	$\partial_T X$
■ $\mathbb{R}^2$		
■ tree T	$\mathcal{C}$	discrete

Proof sketch of Claim 1

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space X	∂X	$\partial_T X$
$\mathbb{R}^2$	$\bigcirc$	$\diamond$
$\mathbb{H}^2$	$\bigcirc$	discrete
tree T	$\mathcal{C}$	discrete