Moufang twin trees and $\mathbb{Z}$-systems

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Motivation: Moufang twin trees

Definition

Let Δ^+, Δ^- be two thick trees.

(1) A symmetric map $\delta : V(\Delta^+) \times V(\Delta^-) \cup V(\Delta^+) \times V(\Delta^-) \to \mathbb{N}$ is called a codistance if

  (i) For $\epsilon \in \{+,-\}, x^\epsilon \in V(\Delta^\epsilon)$ and $y^{-\epsilon}, z^{-\epsilon} \in V(\Delta^{-\epsilon})$ adjacent we have $\delta(x^\epsilon, y^{-\epsilon}) = \delta(x^\epsilon, z^{-\epsilon}) \pm 1$.

  (ii) If $\epsilon \in \{+,-\}$, $x^\epsilon \in V(\Delta^\epsilon)$ and $y^{-\epsilon} \in V(\Delta^{-\epsilon})$ with $\delta(x^\epsilon, y^{-\epsilon}) > 0$, then there exists a unique $z^{-\epsilon} \in V(\Delta^{-\epsilon})$ adjacent to $y^{-\epsilon}$ such that $\delta(x^\epsilon, z^{-\epsilon}) = \delta(x^\epsilon, y^{-\epsilon}) + 1$.

(2) We call $(x^+, y^-) \in V(\Delta^+) \times V(\Delta^-)$ opposite if $\delta(x^+, y^-) = 0$.

(3) If δ is a codistance for Δ^+ and Δ^-, then $(\Delta^+, \Delta^-, \delta)$ is called a twin tree.

Motivation: Moufang twin trees

Definition

Let $\Delta = (\Delta^+, \Delta^-, \delta)$ be a twin tree.

- Let $\alpha^\epsilon = (x_0^\epsilon, x_1^\epsilon, \ldots)$ be a half-apartment (one-direction infinite path) in Δ^ϵ for $\epsilon \in \{+, -\}$. Then (α^+, α^-) is called a twin root if $\delta(x_i^+, x_j^-) = i + j$ for all $i, j \geq 0$.
- Let $\Sigma^\epsilon = (x_n^\epsilon)_{n \in \mathbb{Z}}$ be an apartment (two-direction infinite path) in Δ^ϵ for $\epsilon \in \{+, -\}$. Then (Σ^+, Σ^-) is called a twin apartment if $\delta(x_i^+, x_j^-) = |i - j|$ for all $i, j \in \mathbb{Z}$.
- Let $\alpha = (\alpha^+, \alpha^-)$ be a twin root with $\alpha^\epsilon = (x_n^\epsilon)_{n \geq 0}$ for $\epsilon \in \{+, -\}$. Set $U_\alpha := \{g \in \mathrm{Aut}\Delta \mid y^g = y$ for all y adjacent to some x_n^ϵ for some $n > 0\}$. Then U_α is called the root group associated to α.
- Δ is called Moufang if for every twin root α the root group U_α acts transitively on the set of twin apartments containing α.

Example

- Bruhat-Tits trees for Chevalley groups of relative rank one over $\mathbb{k}((t))$ resp. $\mathbb{k}((t^{-1}))$, where $\mathbb{k}$ is a field.
- Twin trees for Kac-Moody-groups.
- Lot of other examples.
- Tits: There are uncountably many non-isomorphic twin trees of valency 3 (and also for every other valency).

From Moufang twin trees to $\mathbb{Z}$-systems

Let $\Delta = (\Delta^+, \Delta^-, \delta)$ be a Moufang twin tree, $G \leq \mathrm{Aut}\Delta$ containing all root groups and $\Sigma = (\Sigma^+, \Sigma^-)$ a twin apartment with $\Sigma^\epsilon = (x_n^\epsilon)_{n \in \mathbb{Z}}$ and $\delta(x_n^+, x_m^-) = |n - m|$ for all $n, m \in \mathbb{Z}$. Then $\alpha_n := (\alpha_n^+, \alpha_n^-)$ with $\alpha_n^+ = (x_i^+)_{i \geq n}$ and $\alpha_n^- = (x_i^-)_{i \leq n}$ is a twin root for all $n \in \mathbb{Z}$. Set $U_n := U_{\alpha_n}$ and $U := \langle U_n \mid n \in \mathbb{Z} \rangle$ (the unipotent horocyclic group). Then we have:

(i) $[U_m, U_n] \leq U_{m+1} \dots U_{n-1}$ for all $m < n \in \mathbb{Z}$.

(ii) For every $u \in U^*$ there are uniquely determined $m < n \in \mathbb{Z}$, $u_i \in U_i$ for $m \leq i \leq n$ such that $u = u_m \dots u_n$ and $u_m, u_n \neq 1$.

(iii) There is an automorphism $\sigma \in G_\Sigma$ such that $U_n^\sigma = U_{n+2}$ for all $n \in \mathbb{Z}$.

(iv) If $T \leq \mathrm{Aut}\Delta$ is the group fixing Σ pointwise, then $T \leq N_G(U_n)$ for all $n \in \mathbb{Z}$.

This motivates the definition of a $\mathbb{Z}$-system

$\mathbb{Z}$-systems

Definition

Let X be a group, $(X_n)_{n\in\mathbb{Z}}$ be a family of subgroups of X, $\sigma \in \mathrm{Aut}(X)$ and $T \leq \mathrm{Aut}(X)$ with $T^\sigma = T$.

(1) $\big(X, (X_n)_{n\in\mathbb{Z}}, \sigma, T\big)$ is called a $\mathbb{Z}$-system if
   - (i) $[X_m, X_n] \leq X_{m+1} \dots X_{n-1}$ for all $m < n \in \mathbb{Z}$.
   - (ii) For every $x \in X^*$ there are uniquely determined $m < n \in \mathbb{Z}$, $x_i \in X_i$ for $m \leq i \leq n$ such that $x = x_m \dots x_n$ and $x_m, x_n \neq 1$.
   - (iii) $X_n^\sigma = X_{n+2}$ for all $n \in \mathbb{Z}$.
   - (iv) $X_n^h = X_n$ for all $h \in T$ and $n \in \mathbb{Z}$.

(2) A $\mathbb{Z}$-system $\big(X, (X_n)_{n\in\mathbb{Z}}, \sigma, T\big)$ is called irreducible if T acts irreducibly on X_n for all $n \in \mathbb{Z}$.

(3) A $\mathbb{Z}$-system $\big(X, (X_n)_{n\in\mathbb{Z}}, \sigma, T\big)$ is called nilpotent (resp. abelian, etc.) if X_n is nilpotent (resp. abelian etc.) for all $n \in \mathbb{Z}$.

Remark

(1) The map σ is called a shift of lenght 2, the group T the torus.

(2) An irreducible, nilpotent $\mathbb{Z}$-system is abelian.

(3) Not every $\mathbb{Z}$-systems comes from a Moufang twin tree. In a Moufang twin tree, every root groups U_n are isomorphic to a root group of a Moufang set. The proper Moufang sets are conjectured to have nilpotent root groups. Therefore, it is reasonable to focus on nilpotent $\mathbb{Z}$-system.

Example

(i) Let $G := \mathrm{SL}_2(\Bbbk[t,t^{-1}])$,

$$\begin{array}{rcl} U & := & \left\{ \begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} ; f \in \Bbbk[t,t^{-1}] \right\}, \\ U_n & := & \left\{ \begin{pmatrix} 1 & 0 \\ a \cdot t^n & 1 \end{pmatrix} ; a \in \Bbbk \right\} \text{ for } n \in \mathbb{Z}, \\ T & := & \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} ; a \in \Bbbk^* \right\}, \\ \sigma & := & \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}. \end{array}$$

This $\mathbb{Z}$-system comes from the Moufang twin tree of G. It is irreducible unless $\Bbbk$ is a non-perfect field of characteristic 2. We have $[U_n, U_m] = 1$ for all $n, m \in \mathbb{Z}$, hence U is abelian.

Example

(ii) Let $\Bbbk$ be a field with $\operatorname{char}\Bbbk \neq 2$, $\mathbb{K} := \Bbbk(t)$, $* \in \mathrm{Aut}(\mathbb{K})$ with $a^* = a$ for all $a \in \Bbbk$ and $t^* = -t$. Consider the following subgroups of $\mathrm{SL}_3(\Bbbk[t, t^{-1}])$:

$$U := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & -a^* & 1 \end{pmatrix} ; a, b \in \Bbbk[t, t^{-1}], N(a) + \mathrm{Tr}(b) = 0 \right\},$$

$$U_{2n+1} := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b \cdot t^{2n+1} & 0 & 1 \end{pmatrix} ; b \in \Bbbk \right\} \text{ and}$$

$$U_{2n} := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a \cdot t^n & 1 & 0 \\ \frac{a^2}{2} \cdot t^{2n} & a \cdot (-t)^n & 1 \end{pmatrix} ; a \in \Bbbk \right\} \text{ for } n \in \mathbb{Z}.$$

Example

This $\mathbb{Z}$-systems comes from the twin tree for $\mathrm{SU}_3(\Bbbk[t, t^{-1}])$. We have $[U_{4n+2}, U_{4m}] = U_{2n+2m+1}$ for all $n, m \in \mathbb{Z}$ and $[U_k, U_\ell] = 1$ for all other integers k and ℓ. Hence we have $Z(U) = U' = \langle U_{2n+1} \mid n \in \mathbb{Z} \rangle$, thus U is nilpotent of class 2.

Example

(iii) Let A and B be two arbitrary groups. Set
$X := \{x = (x_n)_{n\in\mathbb{Z}} \mid x_{2n} \in A \text{ and } x_{2n+1} \in B \text{ for all } n \in \mathbb{Z}\}$.
$X_n := \{x \in X \mid x_i = 1 \text{ for all } i \neq n\}$,
$\sigma : X \to X : (x_n)_{n\in\mathbb{Z}} \mapsto (x_{n-2})_{n\in\mathbb{Z}}$
Then $\left(X, (X_n)_{n\in\mathbb{Z}}, \sigma, T\right)$ is $\mathbb{Z}$-system.

Theorem

(G., Horn, Mühlherr 2016) Let $(X, (X_n)_{n\in\mathbb{Z}}, \sigma, T)$ be a $\mathbb{Z}$-system such that X_n has prime order for all $n \in \mathbb{Z}$. Then X is nilpotent of class at most 2.

Remark

(1) The proof contains a gap which we were able to fix.

(2) The theorem also follows by a recent result of Glöckner and Willis about contraction groups.

(3) We used another definition of $\mathbb{Z}$-system for this theorem (without T).

(4) The proof relies on the fact that X_n has no proper subgroup. Idea: Consider $\mathbb{Z}$-systems such that X_n has no non-trivial T-invariant subgroup, i.e. irreducible $\mathbb{Z}$-systems.

Definition

Let $(X, (X_n)_{n\in\mathbb{Z}}, \sigma, T)$ be a $\mathbb{Z}$-system

(i) For $m \le n \in \mathbb{Z} \cup \{\infty, -\infty\}$ set $X_{m,n} := \langle X_i \mid m \le i \le n\rangle$.

(ii) For $x \in X^*$ set $\mu(x) := \max\{m \in \mathbb{Z} \mid x \in X_{m,\infty}\}$, $\nu(x) := \min\{n \in \mathbb{Z} \mid x \in X_{-\infty,n}\}$ and $\omega(x) := n - m \ge 0$.

(iii) Call $x \in X^*$ even (resp. odd) if $\mu(x)$ is even (resp. odd).

(iv) A subgroup $Y \le X$ is called shift-invariant if $Y^\tau = Y$ for all $\tau \in \langle T, \sigma\rangle$.

(v) We say that a subgroup Y of X has finite T-index if there exists a finite subset M of X with $X = \langle Y \cup M^T\rangle$. Otherwise, Y has infinite T-index.

Shift-invariant subgroups

Remark

(1) If $Y, Z \leq X$ are shift-invariant subgroups, then $\langle Y, Z\rangle$, $Y \cap Z$ and $[Y, Z]$ are also shift-invariant.

(2) If $Y \leq X$ is shift-invariant and Z is characteristic in Y, then Z is also shift-invariant. In particular, every characteristic subgroup of X is shift-invariant.

(3) If X_n is finite for all $n \in \mathbb{Z}$, then $Y \leq X$ has finite T-index if and only if $|X : Y| < \infty$.

Shift-invariant subgroups

From now on, $\big(X, (X_n)_{n\in\mathbb{Z}}, \sigma, T\big)$ is irreducible and nilpotent (hence abelian).

Theorem

Let $1 \neq Y \trianglelefteq X$ be shift-invariant.

(i) *Y is of finite T-index if and only if Y contains both odd and even elements.*

(ii) *If Y is of infinite T-index and $y \in Y^*$ with $\omega(y)$ minimal, then $Y = \langle y^{\langle T,\sigma\rangle}\rangle$.*

(iii) *If Y is of finite T-index and $x, y \in Y^*$ even resp. odd with $\omega(x), \omega(y)$ minimal, then $Y = \langle x^{\langle T,\sigma\rangle} \cup y^{\langle T,\sigma\rangle}\rangle$.*

Shift-invariant subgroups

Theorem

Let $1 \neq Y \trianglelefteq X$ be shift-invariant. Then $Y' < Y$.

Theorem

X' has infinite T-index.

Theorem

Let $Y \trianglelefteq X$ be shift-invariant and of infinite T-index. Then $[Y, X] = \big[[Y, X], X\big]$.

Theorem

Let $Y \trianglelefteq X$ be shift-invariant with $[Y, X] = Y$. Then Y is abelian.

Shifts of length 1

Corollary

If we have a shift of length 1, i.e. an automorphism τ of X with $T^\tau = T$ and $X_n^\tau = X_{n+1}$ for all $n \in \mathbb{Z}$, then X is abelian.

Proof.

Since X' is shift-invariant and of infinite T-index, X' cannot contain both odd and even elements. But τ normalises X' and interchanges the sets of even and odd elements. Hence $X' = 1$. $\square$

Main result

Theorem

Let $(X, (X_n)_{n\in\mathbb{Z}}, \sigma, T)$ be an irreducible, nilpotent $\mathbb{Z}$-system. Then X is nilpotent of class at most 2.

Proof.

Consider the shift-invariant normal subgroup $Y := [X', X]$. Then Y is abelian. X acts on the abelian group Y by conjugation. We have $[Y, X] = Y$. If $Y \neq 1$, then we find $Z \trianglelefteq Y$ such that X acts trivially on Y/Z, contradiction. Hence $Y = 1$, so $X' \leq Z(X)$. $\square$

Conjecture

Let $(X, (X_n)_{n\in\mathbb{Z}}, \sigma, T)$ *be a nilpotent* $\mathbb{Z}$*-system.*

(1) X *is nilpotent.*

(2) *There is a function* $f : \mathbb{N}^2 \to \mathbb{N}$ *such that the class of* X *is bounded by* $f(n_0, n_1)$*, where* n_i *is the class of* X_i *for* $i = 0, 1$.

THANKS FOR YOUR ATTENTION