Automorphisms groups of Cayley graphs of Coxeter groups

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Motivation

Automorphism groups of locally finite connected graphs naturally possess a totally disconnected locally compact (TDLC) topology, namely the pointwise-convergence/permutation topology.
Furthermore, this group is compactly generated.
And conversely, every compactly generated TDLC group acts transitively on some locally finite graph such that the vertex stabilisers are open compact.
Some questions naturally arise:

Question

Can we give some "sufficiently nice" characterisation of transitive graphs whose automorphism group is not discrete/countable?

These questions are very difficult in general, but if we restrict our attention to some specific class of graphs, they can be answered.

Given an undirected finite graph $\Gamma = (V\Gamma, E\Gamma)$ and $m\colon V\Gamma \times V\Gamma \to \{1, 2, \dots\} \cup \{+\infty\}$ be a function, called a *weight*, satisfying

c_1) $m(x, y) = m(y, x)$ for all $x, y \in V\Gamma$;

c_2) $m(x, y) = 1$ if and only if $x = y$;

c_3) $m(x, y) = +\infty$ if and only if $\{x, y\} \notin E\Gamma$,

the corresponding *Coxeter group* is given by the presentation

$$W_\Gamma := \langle V\Gamma \mid (xy)^{m(x,y)} = e \quad \forall\, x, y \in V\Gamma \rangle, \tag{1}$$

where $(xy)^{+\infty}$ by definition means that there is no relation between the two generators x and y. We call $\Gamma = (V\Gamma, E\Gamma, m)$ a weighted graph. (Not to be confused with Dynkin diagrams!!!)

The main result

Theorem (Berlai, MF)

Let Γ be a finite weighted graph and let $\mathcal{C}_\Gamma$ be the Cayley graph of the Coxeter group W_Γ with respect to the standard generating set $V\Gamma$. The automorphism group $\mathcal{A}_\Gamma$ is non-discrete if and only if there exists $x \in V\Gamma$ and a non-trivial automorphism $\alpha \in \mathrm{Aut}(\Gamma)$ such that $\alpha \restriction_{\mathrm{star}(x)} = \mathrm{id}_\Gamma$.
Furthermore, if $\mathcal{A}_\Gamma$ is discrete, then

$$\mathcal{A}_\Gamma \simeq W_\Gamma \rtimes \mathrm{Aut}(\Gamma).$$

Compare that with the case of right-angled Artin groups (*RAAGs*).

Theorem (Taylor)

Let Γ be a finite graph and let $\mathcal{C}_\Gamma$ be the Cayley graph of the RAAG A_Γ with respect to the standard generating set $V\Gamma$. The automorphism group $\mathcal{A}_\Gamma$ is non-discrete if and only if Γ is not complete.

Local actions

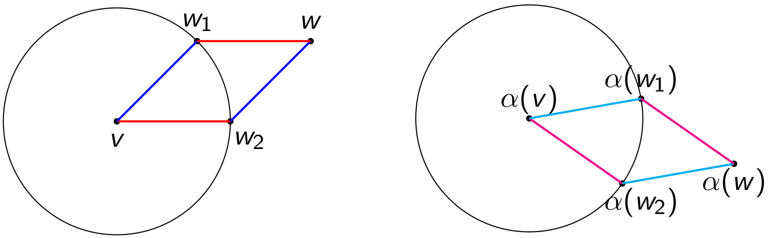
An automorphism $\alpha \in \mathcal{A}_\Gamma$ and a vertex v in the Cayley graph $\mathcal{C}_\Gamma$ uniquely define an automorphism $\sigma(\alpha, v) \in \mathrm{Aut}(\Gamma)$, as follows. As α is an automorphism, it induces a bijection between the edges incident to v and the edges incident to $\alpha(v)$, and therefore a bijection $\sigma(\alpha, v) \colon V\Gamma \to V\Gamma$ between their labels. This map is an automorphism of the weighted graph Γ, that is, it preserves the weights $m(x, y)$ for all $x, y \in V\Gamma$.

Lemma

Let $\alpha \in \mathcal{A}_\Gamma$, let v be a vertex in $\mathcal{C}_\Gamma$ and $x \in V\Gamma$. Then

$$\sigma(\alpha, v) \restriction_{\mathrm{star}(x)} = \sigma(\alpha, vx) \restriction_{\mathrm{star}(x)} .$$

Proof by handwaving



(Almost) translations

Let Γ be a weighted graph and consider the Cayley graph $\mathcal{C}_\Gamma$ of the associated Coxeter group. We say that $\alpha \in \mathcal{A}_\Gamma$ is

1. a *translation* if it is label-preserving (in the sense of Cayley graphs), that is for all edges $\{v, w\}$ in $\mathcal{C}_\Gamma$ the edges $\{v, w\}$ and $\{\alpha(v), \alpha(w)\}$ have the same label;
2. an *almost translation* if $\sigma(\alpha, v) = \sigma(\alpha, w)$ for all vertices v, w in $\mathcal{C}_\Gamma$.

Lemma

The subgroup of $\mathcal{A}_\Gamma$ consisting of translations is isomorphic to W_Γ and the subgroup of $\mathcal{A}_\Gamma$ generated by almost translations is isomorphic to $W_\Gamma \rtimes \mathrm{Aut}(\Gamma)$, and in particular it is finitely generated.

Proof by hand waving

1. We can identify W_Γ with the subgroup of $\mathcal{A}_\Gamma$ consisting of translations.
2. We can identify $\mathrm{Aut}(\Gamma)$ with the subgroup of $\mathcal{A}_\Gamma$ consiting of almost translations that fix the origin.
3. we check that
   - $W_\Gamma \cap \mathrm{Aut}(\Gamma) = \{1\}$;
   - $\langle W_\Gamma, \mathrm{Aut}(\Gamma)\rangle \leq \mathcal{A}_\Gamma$ contains all almost translations;
   - $W_\Gamma \trianglelefteq \langle W_\Gamma, \mathrm{Aut}(\Gamma)\rangle$.

Separating sets

Definition

Let $S \subsetneq \Gamma$ be a proper separating set, so that $\Gamma \setminus S = C_1 \sqcup \cdots \sqcup C_n$ is the disjoint union of $n \geqslant 2$ connected components. Suppose that there exists a non-empty set $I \subsetneq \{1, \ldots, n\}$ with the property that, fixing $\Gamma_1 := S \sqcup \bigsqcup_{i \in I} C_i$ and $\Gamma_2 := \bigsqcup_{i \notin I} C_i$, there exists a non-trivial $\alpha \in \mathrm{Aut}(\Gamma_1)$ such that $\alpha \restriction_S = \mathrm{id}_S$.
Then we call S a *good separating set*.

Lemma

Let Γ be a finite simplicial graph. The following are equivalent:

(1) *there exists a vertex $v \in V\Gamma$ and a non-trivial $\alpha \in \mathrm{Aut}(\Gamma)$ such that $\alpha \restriction_{\mathrm{star}(v)} = \mathrm{id}_\Gamma$;*

(2) *Γ admits a good separating set;*

(3) *there exist a separating set S and two distinct elements $\alpha, \beta \in \mathrm{Aut}(\Gamma_1)$ such that $\alpha \restriction_S = \beta \restriction_S$.*

Proof by handwaving

(1) $\rightarrow$ (2) If $\alpha\restriction_{\mathsf{star}(v)} = \mathrm{id}_\Gamma$ then star(v) is a good separating set.

(2) $\rightarrow$ (3) If S is a good separating set, then there take $\alpha \in; \mathrm{Aut}(\Gamma) \setminus \{\mathrm{id}_\Gamma\}$ such that $\alpha\restriction_{\mathsf{star}(v)} = \mathrm{id}_\Gamma$ and id_Γ.

(3) $\rightarrow$ (1) Clearly $\alpha^{-1}\beta$ is nontrivial and fixes S pointwise, so S is a good separating set.

Good separating sets are the key

Presence of a good separating set allows us to construct uncountably many distinct automorphisms of $\mathcal{C}_\Gamma$

Lemma (There...)

If Γ has a good separating set then $\mathcal{A}_\Gamma$ is uncountable.

And conversely, the lack thereof allows us to fully describe the automorphism group.

Lemma (...and back again.)

Let Γ be a connected, finite simplicial graph, and suppose that there is no good separating set in Γ. Then all elements in $\mathcal{A}_\Gamma$ are almost translations, that is $\mathcal{A}_\Gamma = W_\Gamma \rtimes \mathrm{Aut}(\Gamma)$. In particular, the automorphism group is finitely generated, and therefore countable.

Proof by handwaving - There...

1. If S is a separating set then W_Γ splits as an amalgamated free product $W_\Gamma = W_{\Gamma_1} *_{W_S} W_{\Gamma_2}$
2. If S is a good separating set then there is nontrivial $\alpha \in \mathrm{Aut}(\Gamma_1)$ such that $\alpha \restriction_S = \mathrm{id}_S$.
3. Decompose $\mathcal{C}_\Gamma$ into connected subspaces corresponding to cosets of W_{Γ_1}. They are connected via cosets of W_S.
4. On each coset gW_{Γ_1} we can decide whether or not to realise α as an almost translation fixing g.

Proof by handwaving - ...and back again

1. Suppose that $\alpha \in \mathcal{A}_\Gamma$ is not an almost translation. Then there are vertices $v, w \in W_\Gamma$ such that $\sigma(\alpha, v) \neq \sigma(\alpha, v)$.
2. WLOG we may assume that v, w are adjacent in W_Γ, so $w = vx$.
3. Then $\mathrm{star}(x)$ must be a good separating subset in Γ!

Thank you!