LG-RIGIDITY OF QUASI-BUILDINGS

Graphs, knitting and embroidery

Amandine Escalier

Magdeburg

October 2021

Plan

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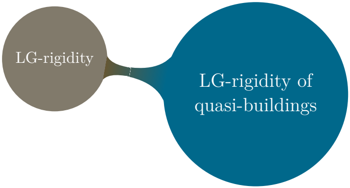

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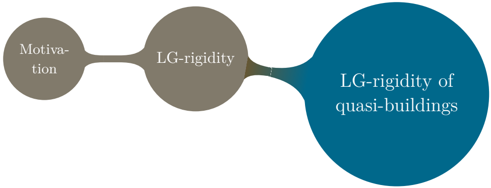

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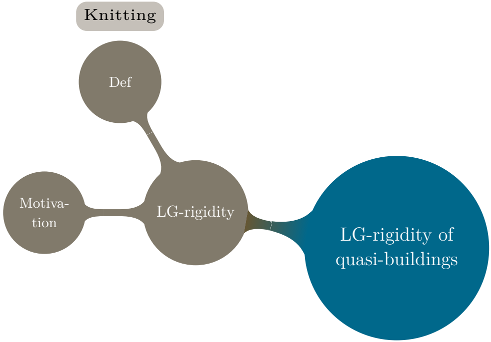
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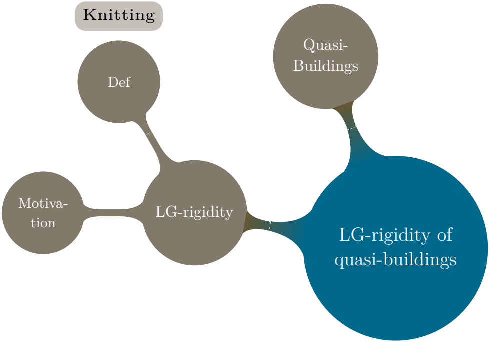
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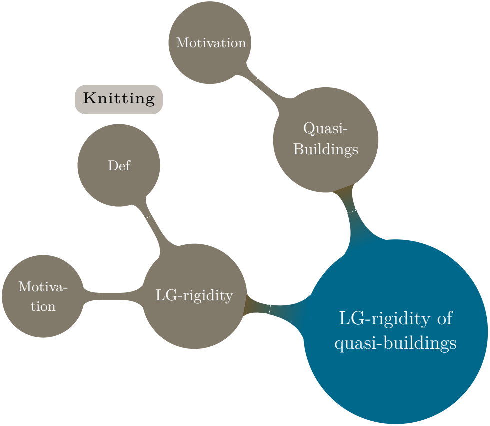

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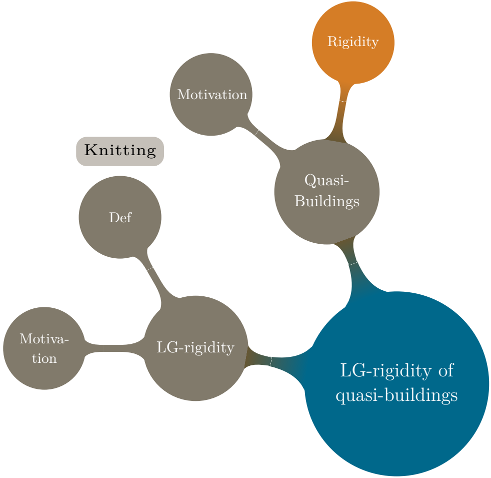
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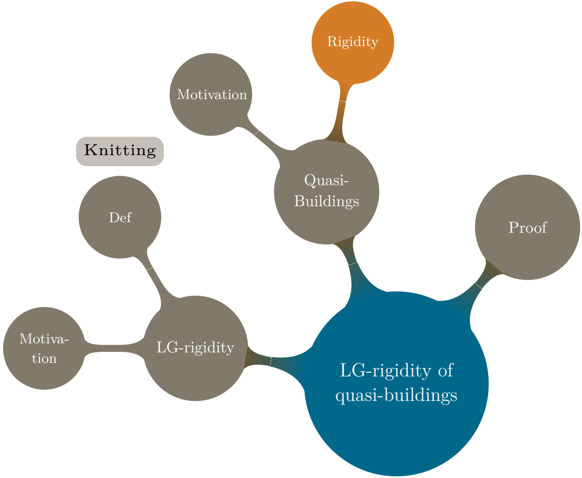

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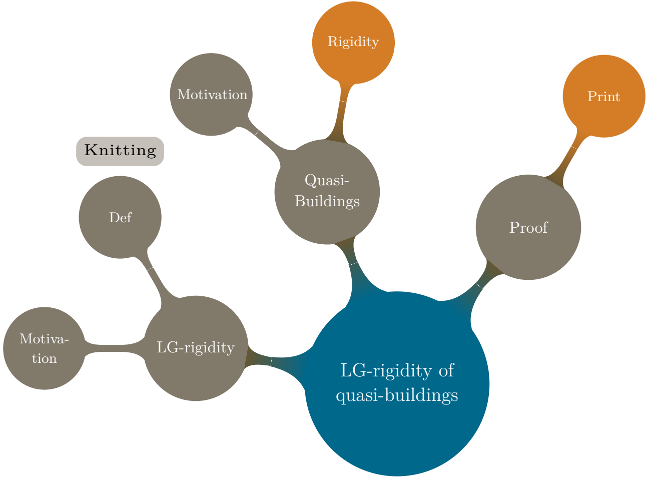
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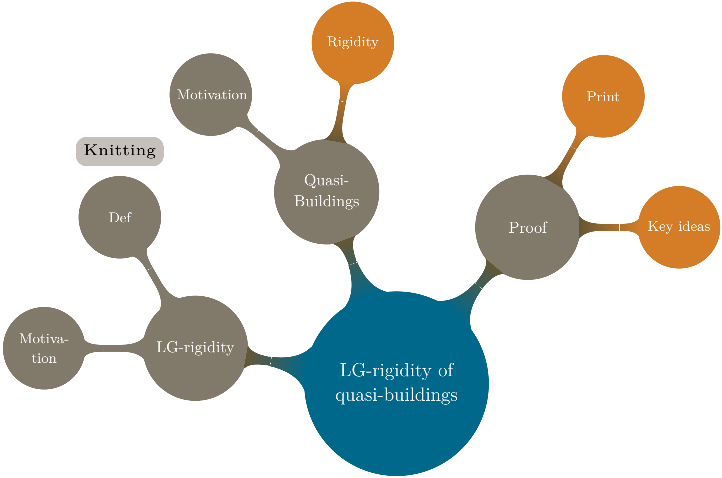
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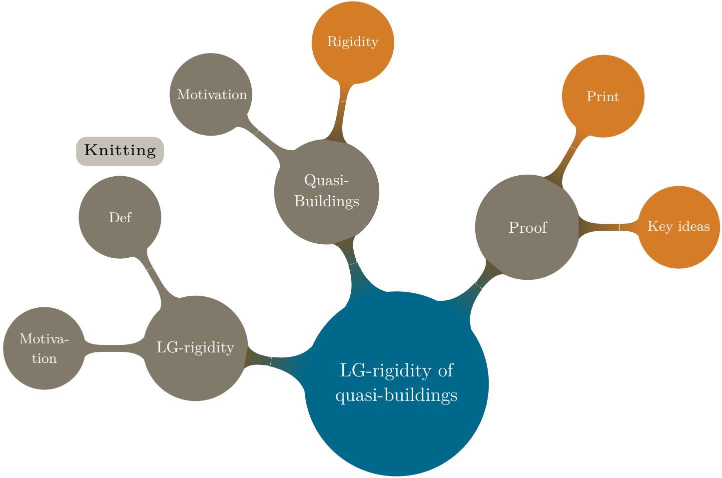
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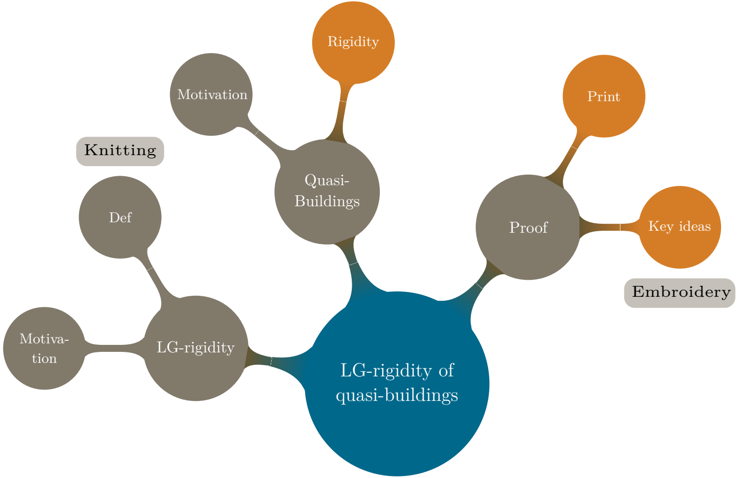

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Embroidery
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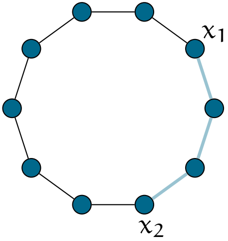
Can we tell the difference between
a scarf and a sweater
knowing only a part of the knitting?

Conventions

- Two vertices linked by an edge are at *distance* 1.
  → It induces a distance on the entire graph.

Exemple

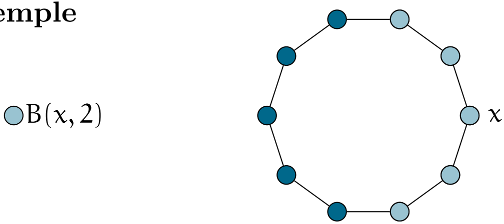
$d(x_1, x_2) = 3$



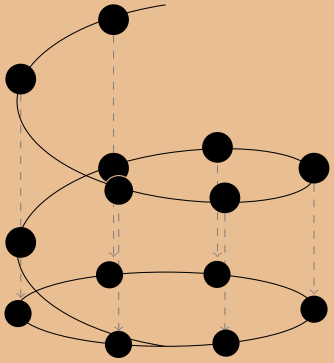
Conventions

- Two vertices linked by an edge are at *distance* 1.
  $\rightarrow$ It induces a distance on the entire graph.
- $B_X(x, R) :=$ {vertices at distance R from x}.

Exemple



I — Local-to-Global rigidity



I.1 – Graphs locally the same

Motivations

- Every regular graph of degree d is covered by the d-regular tree.

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Let $R > 0$ and let X, Y be two graphs.

Definition. We say that X is **R-locally** Y, if:

$$(\forall x \in X)\ (\exists y \in Y)\ : B_X(x, R) \text{ isometric to } B_Y(y, R).$$

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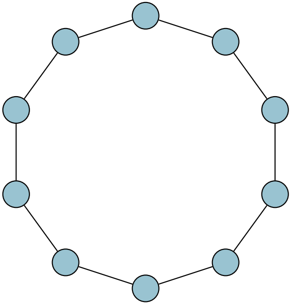

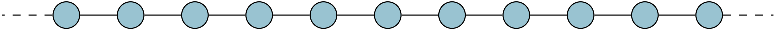

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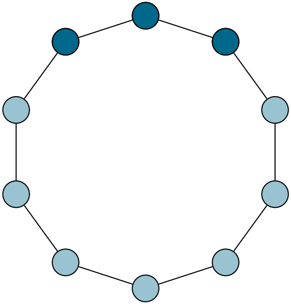

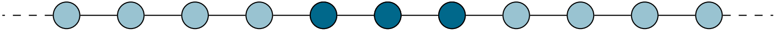

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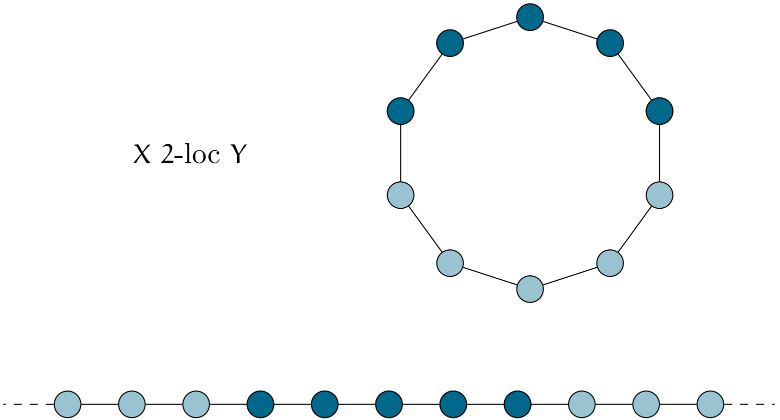

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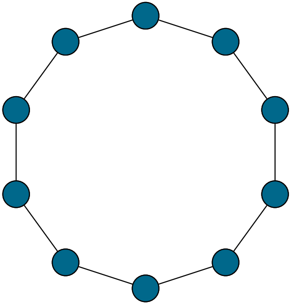

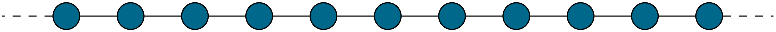

I.2 — Local-to-Global Rigidity

Definition.

- We say that $\rho : X \to Y$ is a **covering** if ρ is surjective and if:

$$\forall x \in X \qquad \rho : B(x,1) \xrightarrow{\text{bijective}} B\big(\rho(x),1\big).$$

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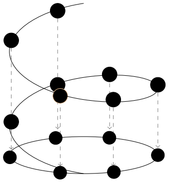
Example

$$\rho : \begin{cases} \mathbb{Z} & \to \mathbb{Z}/6\mathbb{Z}, \\ m & \mapsto m \mod 6. \end{cases}$$



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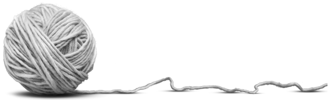
Interlude: Knitted Interpretation



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For all $m > 0$, the Cayley graph $(\mathbb{Z}^m, \{\pm e_1, \dots, \pm e_d\})$ is LG-rigid at scale 3.

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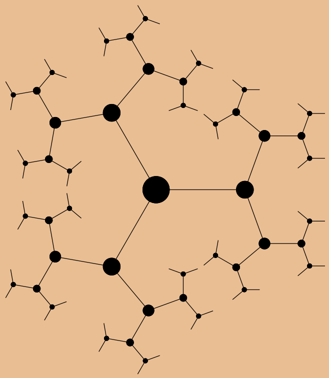
$\to$ *Characteristic* p.

II — From buildings to quasi buildings



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→ What about **quasi-buildings of higher dimension?**

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Theorem. [E., '20] Let $\mathbb{K}$ be a non-Archimedean local field s.t. $\mathsf{char}(\mathbb{K}) = 0$ and $\mathcal{X}$ be the building of $\mathsf{PSL}_n(\mathbb{K})$ with $n \neq 3$. If X is a graph that is **quasi-isometric** to $\mathcal{X}$ and **sufficiently friendly**[1], then X is LG-rigid.

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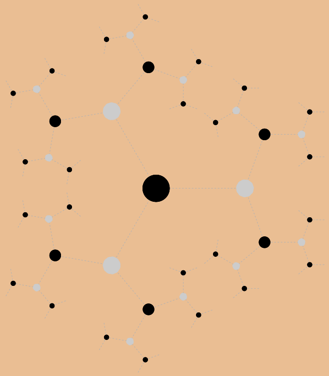
Example Torsion-free lattices of $\mathsf{PSL}_n(\mathbb{K})$ are quasi-isometric to the building and friendly.

III — Idea of the proof



III.1 — Prints

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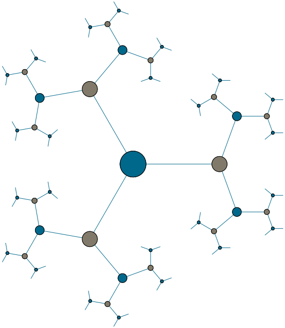
- The vertices can be colored such that **two adjacent vertices have different colors.**

Example (for n=2)



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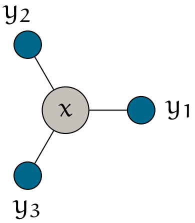
Example in dim. 2



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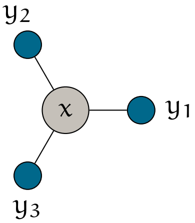

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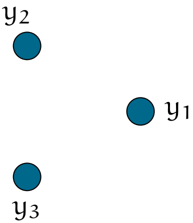

Print of color **blue**

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Property.[E, '20] Let $\mathfrak{c}$ be a color of vertices.
A vertex of the building is uniquely determined by its print of color $\mathfrak{c}$.

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Example In the last example x is uniquely determined by the data $\{y_1, y_2, y_3\}$.

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Example In the last example x is uniquely determined by the data $\{y_1, y_2, y_3\}$.

We only need a **partial** local information to reconstruct the building.

Key Idea Use the building's rigidity

$$\begin{tikzcd} X \\ Y \arrow[u, dashed, "\text{R-loc}"] \end{tikzcd}$$

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quasi-isometric to the building

$$
\begin{array}{ccc}
\mathsf{X} & \xrightarrow{q} & \mathcal{X} \\
\uparrow & & \\
\mathsf{Y} & &
\end{array}
$$

R-loc

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- **Step 1** Reconstruct (locally) the building;

$$\begin{tikzcd} \mathsf{X} \arrow[r, "q"] & \mathcal{X} \\ \mathsf{Y} \arrow[u, dashed] \end{tikzcd}$$

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R-loc

Construction of a graph locally $\mathcal{X}$

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"Bricks" coming from Y, assembled on the model of $\mathcal{X}$

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$$
\begin{tikzcd}
\mathsf{X} \arrow[r, "q"] & \mathcal{X} \arrow[d, "\iota"] \\
\mathsf{Y} \arrow[u, dashed] & \mathcal{Y}
\end{tikzcd}
$$

R-loc

Covering

Use of the building's LG-Rigidity

Key Idea Use the building's rigidity

- **Step 1** Reconstruct (locally) the building;
- **Step 2** Pullback the covering.

R-loc

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\begin{tikzcd}
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"Pullback" the covering by $\mathcal{X}$ to X

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"Pullback" the covering by $\mathcal{X}$ to X

III.3 — Edges embroidery: General idea

Definition of edges in $\mathcal{Y}$

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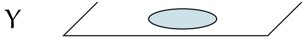
III.3 — Reconstructing the building's vertices



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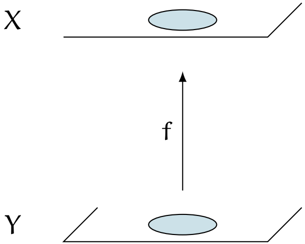

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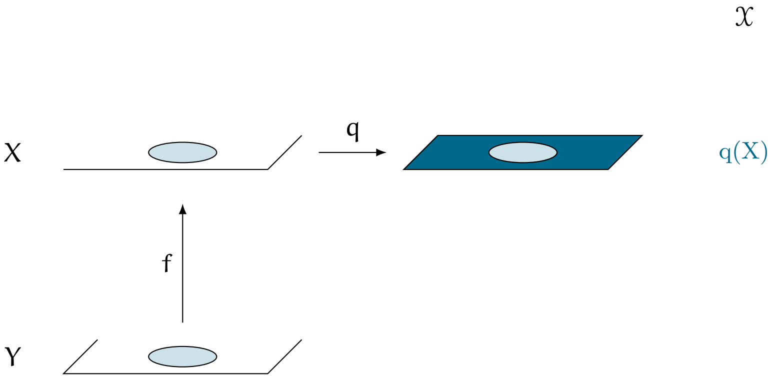

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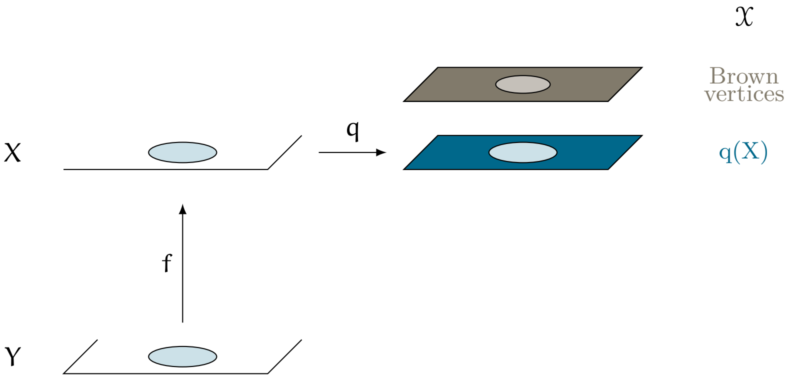

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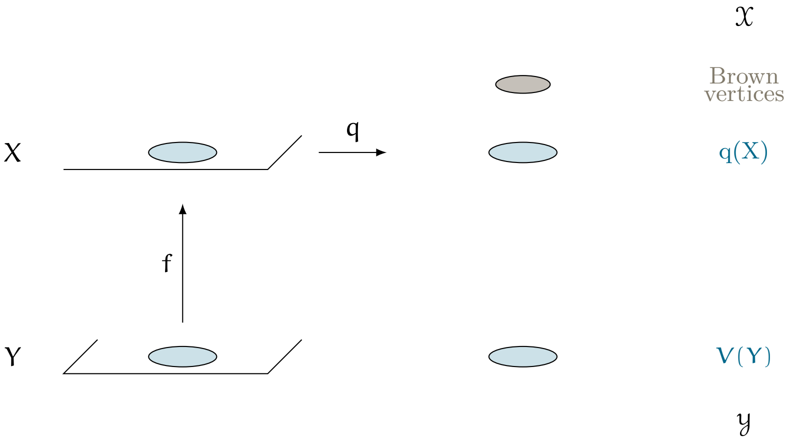

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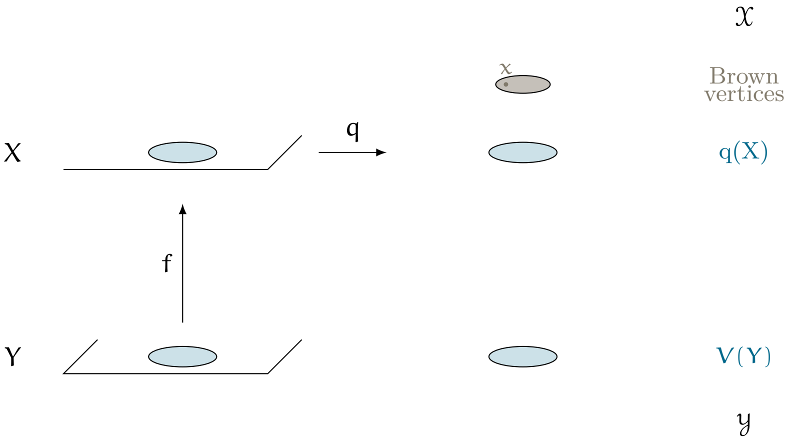

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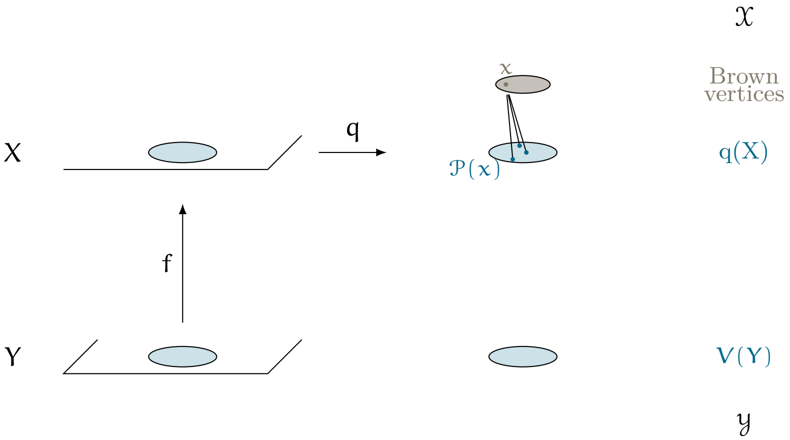

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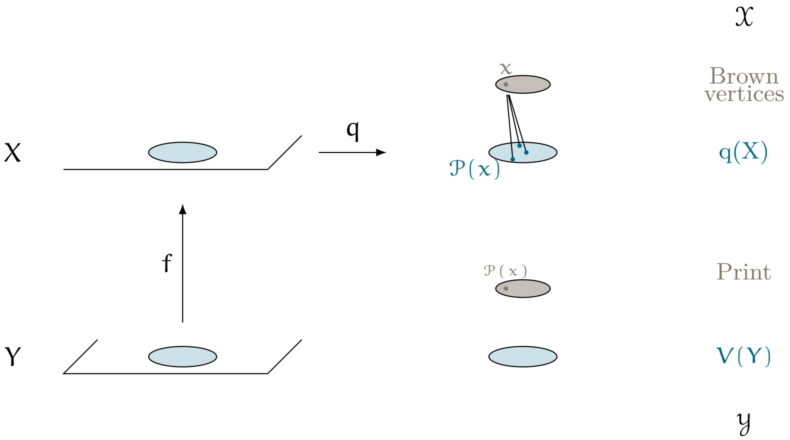

Conclusion — Summary

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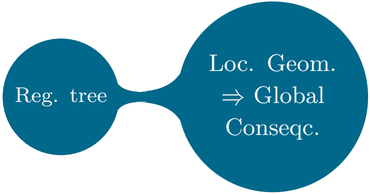

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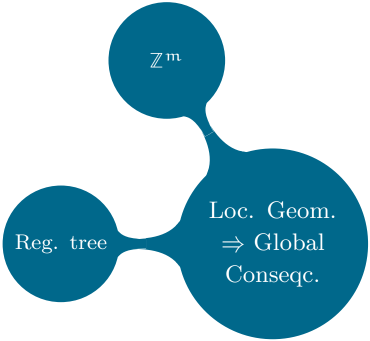

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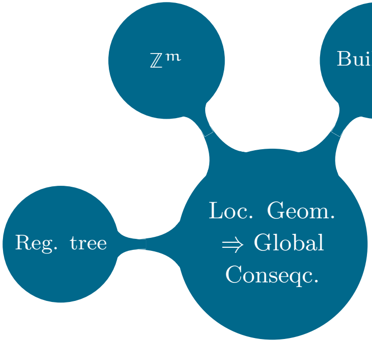

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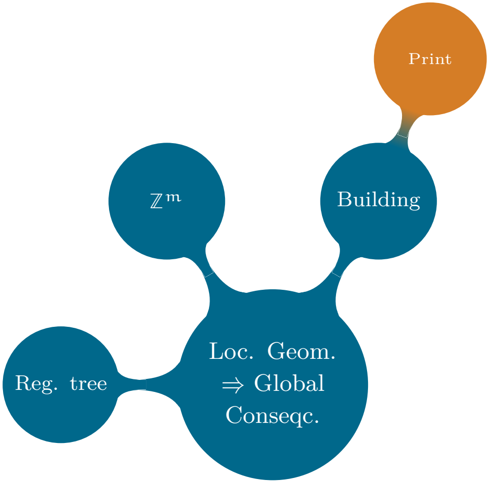

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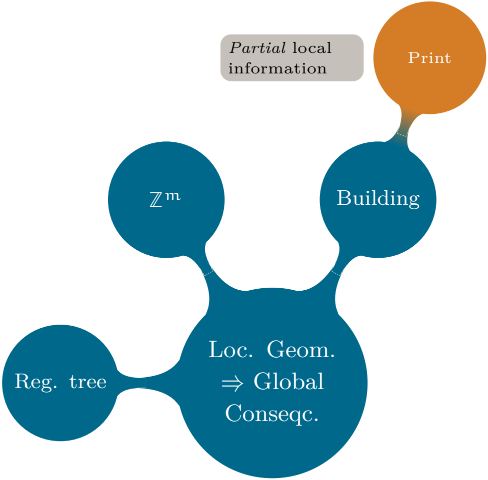

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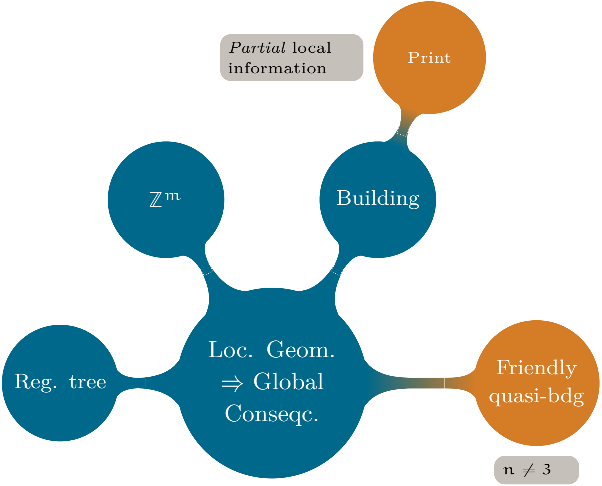

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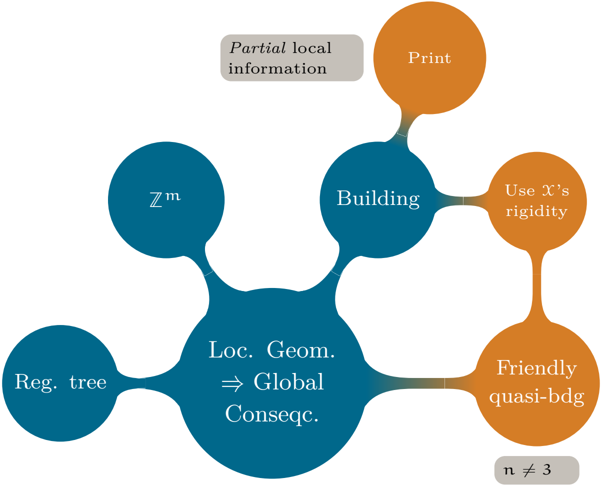

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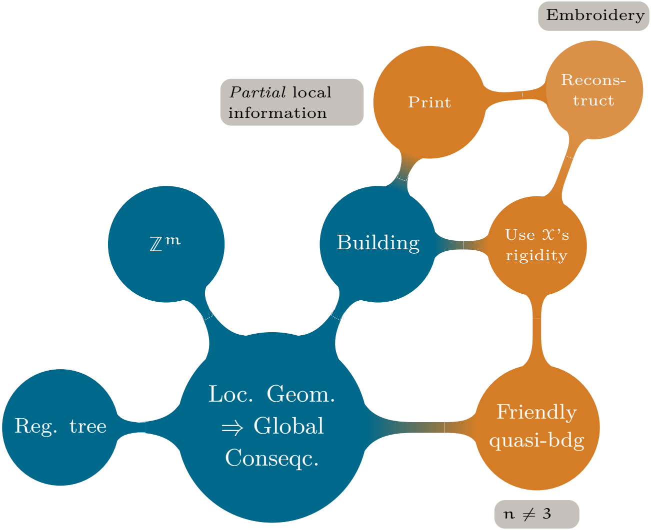

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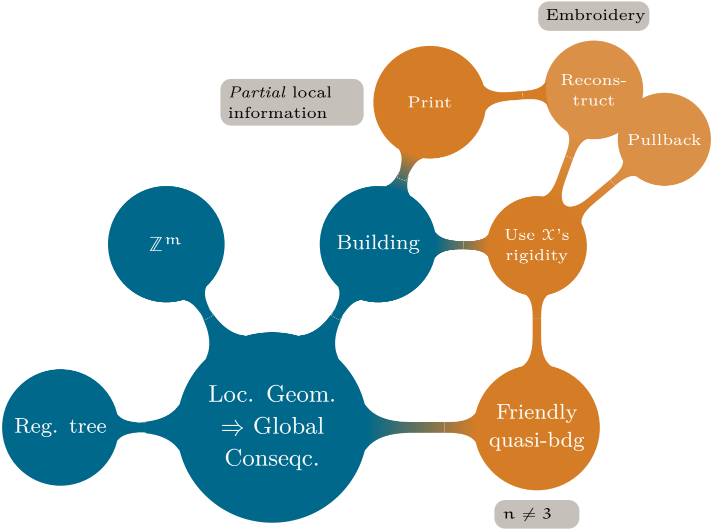

Thank you
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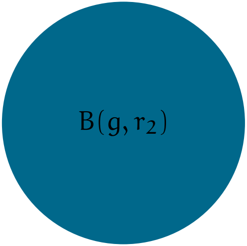

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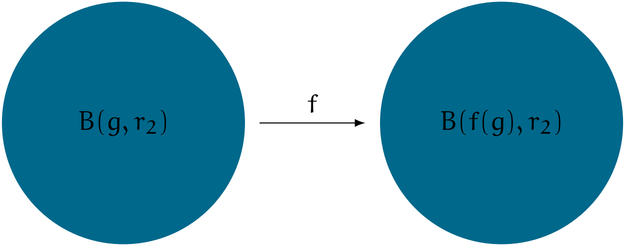

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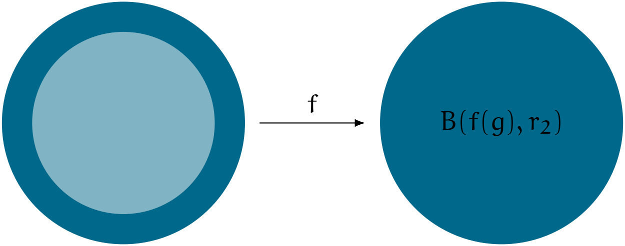

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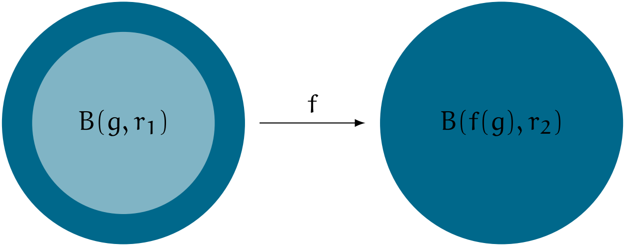

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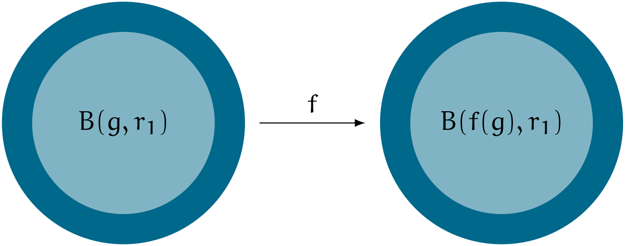

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$\rightarrow$ The edges are defined on the entire graph $\mathcal{Y}$.

Local maps & Planisphere

Fix $y \in Y$ and $f : B_Y(y, R) \to B_X(f(y), R)$.

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To each **local map f** corresponds a **planisphere** $q^{-1}\iota_f \kappa$ centered in y.

Overview

$$
\begin{tikzcd}
X & \mathcal{X} \arrow[l, "q^{-1}"'] \\
Y \arrow[r, "\kappa"'] & \mathcal{Y} \arrow[u, "\iota_f"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
B_{\mathcal{X}}\big(qf(y), r\big) \\
B_{\mathcal{Y}}(y, r) \arrow[u, "\phi_f = \iota_f"']
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\end{tikzcd}$$

Goal: Show that $q^{-1}\iota_f\kappa$ is an isometry from Y to X.

Method

- The map $\mathsf{q}^{-1}\iota_{\mathsf{f}}\kappa$ is **bijective** from Y to X.

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$\Rightarrow$ **Global isometry**.