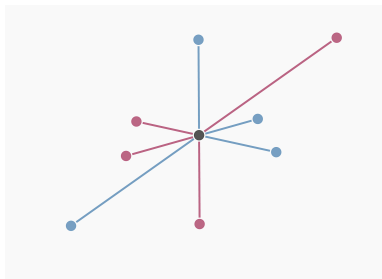


Orders and Polytopes: Matrix Algebras from Valuations

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TR 195
SYMBOLIC TOOLS

Max-Planck-Institut für

Mathematik

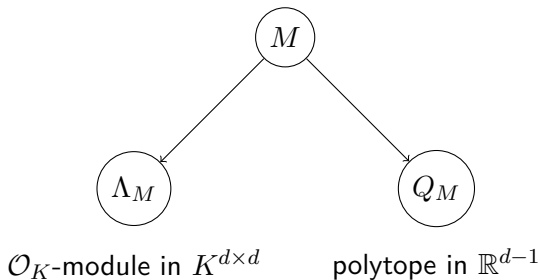
in den **Naturwissenschaften**

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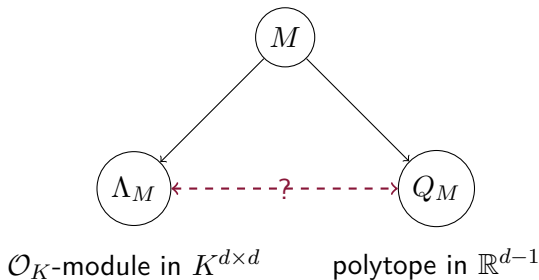


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Goal: understand what the interplay between these objects is.

Pioneers: Plesken and Zassenhaus (1983).



Let K be a field with a surjective valuation map

$$\text{val} : K \rightarrow \mathbb{Z} \cup \{\infty\}.$$

Example. $K = \mathbb{Q}$ and $\text{val} = \text{val}_3$ is the 3-adic valuation. Then

$$\text{val}(45) = \text{val}(3^2 5) = 2, \quad \text{val}(13/27) = \text{val}(3^{-3} 13) = -3.$$

Denote

- $\mathcal{O}_K = \{x \in K : \text{val}(x) \geq 0\}$ is the **valuation ring** of K ,
- $\mathfrak{m}_K = \{x \in K : \text{val}(x) > 0\} \triangleleft \mathcal{O}_K$ unique maximal,
- $\pi \in K$ such that $\text{val}(\pi) = 1$ is a **uniformizer**.

Then $\mathfrak{m}_K = (\pi)$ and every ideal of \mathcal{O}_K is of the form $\mathfrak{m}_K^k = (\pi^k)$.
If $\text{val} = \text{val}_p$ is the p -adic valuation, then $\pi = p$.



The valuation val can be extended to K^d or $K^{d \times d}$ coordinate-wise:

$$\text{val}_3(2, 15, -1/12) = (0, 1, -1), \quad \text{val}_3 \begin{pmatrix} 0 & 66 \\ -1/3 & 7 \end{pmatrix} = \begin{pmatrix} \infty & 1 \\ -1 & 0 \end{pmatrix}$$



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Then the set

$$\Lambda_M = \{X \in K^{d \times d} : \text{val}(X) \geq M\}$$

is an \mathcal{O}_K -module because in K :

- $\text{val}(x + y) \geq \min\{\text{val}(x), \text{val}(y)\}$
- $\text{val}(xy) = \text{val}(x) + \text{val}(y)$



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Remark. Λ_M has maximal rank d^2 as a free \mathcal{O}_K -submodule of $K^{d \times d}$ and it lives in a ring!



Example. For $K = \mathbb{Q}$, $d = 3$, and $\text{val} = \text{val}_p$:

$$\underbrace{\text{val}_p \begin{pmatrix} 1 & 1 & p \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_X = \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_M \text{ but } \begin{pmatrix} \star & \star & 0 \\ \star & \star & \star \\ \star & \star & \star \end{pmatrix} = \underbrace{\text{val} \begin{pmatrix} 2+p & 2+p & 1+2p \\ 3 & 3 & 2+p \\ 3 & 3 & 2+p \end{pmatrix}}_{X^2}$$

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Example. If $M = \mathbf{0}_{d \times d}$, then $\Lambda_M = \mathcal{O}_K^{d \times d}$ is a ring!



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so Λ_M is not a ring.

Example. $M = \mathbf{0}_{d \times d} \Rightarrow \Lambda_M = \mathcal{O}_K^{d \times d}$ is a **maximal order** in $K^{d \times d}$!

Definition.

- A **(\mathcal{O}_K -)lattice** in K^d is a free \mathcal{O}_K -submodule of rank d .
- An **order** in $K^{d \times d}$ is a lattice in $K^{d \times d}$ that is also a ring.

Remark. Orders of the form Λ_M are called **graduated**, tiled, split or monomial by different authors.



Proposition (Plesken). Λ_M is an order if and only if

$$m_{ii} = 0, \quad m_{ij} + m_{jk} \geq m_{ik} \quad \text{for } 1 \leq i, j, k \leq d. \quad (1)$$

Remark. If $m_{ii} = 0$, we write $M \in \mathbb{Z}_0^{d \times d}$. From now on we will often implicitly assume that (1) holds.



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Remark. Since $\Lambda_M \subseteq K^{d \times d}$, it naturally operates on K^d . We are interested in the lattices that are Λ_M -stable, i.e. all L such that $\Lambda_M L \subseteq L$. Write $L_0 = \mathcal{O}_K e_1 \oplus \dots \oplus \mathcal{O}_K e_d$ for the **standard lattice** (wrt standar basis).



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Example.

- The $\mathcal{O}_K^{d \times d}$ -stable lattices are $\{\pi^n L_0 : n \in \mathbb{Z}\}$. (maximality)
- If L, L' are Λ_M -stable, then $L \cap L'$ and $L + L'$ are Λ_M -stable.



Proposition (Plesken). A lattice L in K^d is stable under Λ_M if and only if there exists $u \in \mathbb{Z}^d$ with

$$u_i - u_j \leq m_{ij} \quad \text{for } 1 \leq i, j \leq d, \quad (2)$$

such that $L = L_u = \mathcal{O}_K \pi^{u_1} e_1 \oplus \dots \oplus \mathcal{O}_K \pi^{u_d} e_d$. Moreover, two stable lattices L_u and $L_{u'}$ are isomorphic as Λ_M -modules if and only if there exists $n \in \mathbb{Z}$ such that $L_{u'} = \pi^n L_u$.



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Remark. Let $u \in \mathbb{Z}^d$ and set $M(u) = (u_i - u_j)$.

- Lattices with $L' = \pi^n L$ are called **equivalent**, denoted $L \sim L'$
- Lattices of the form L_u are called **diagonal** and all have **compatible bases**
- $\text{End}_{\mathcal{O}_K}(L_u) = \Lambda_{M(u)} = \{X \in K^{d \times d} : \text{val}(X) \geq M(u)\}$



Let $\Gamma = \{L_1, \dots, L_n\}$ be a finite set of lattices in K^d . Then the Plesken-Zassenhaus ring

$$\text{PZ}(\Gamma) = \text{End}_{\mathcal{O}_K}(L_1) \cap \dots \cap \text{End}_{\mathcal{O}_K}(L_n)$$

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Remark.

- Each L_i is $\text{PZ}(\Gamma)$ -stable.
- Any $L_i \cap L_j$ and $L_i + L_j$ is $\text{PZ}(\Gamma)$ -stable.
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Proposition (Plesken). If Γ consists of diagonal lattices, then $\text{PZ}(\Gamma)$ is graduated. Moreover, the converse is also true.



The **min-plus** and **max-plus algebras** $(\mathbb{R}, \underline{\oplus}, \odot)$ and $(\mathbb{R}, \overline{\oplus}, \odot)$ are defined by the operations

$$a \underline{\oplus} b = \min\{a, b\}, \quad a \overline{\oplus} b = \max\{a, b\}, \quad a \odot b = a + b.$$

Example. $L_u \cap L_{u'} = L_{u \overline{\oplus} u'}$ and $L_u + L_{u'} = L_{u \underline{\oplus} u'}$



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These operations induce also product of matrices.

Example. If $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $N = \begin{pmatrix} -2 & 0 \\ 3 & 1 \end{pmatrix}$, and $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ then

$$\begin{aligned} M \overline{\odot} N &= \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}, & M \overline{\odot} u &= \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \\ M \underline{\odot} N &= \begin{pmatrix} -2 & 0 \\ -1 & 1 \end{pmatrix}, & M \underline{\odot} u &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$



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Proposition.

- Λ_M is an order if and only if $M \underline{\odot} M = M$
- L_u is a stable lattice if and only if $M \underline{\odot} u^t \geq u^t$
- L_u is Λ_M -stable iff $[u] \in Q_M \cap (\mathbb{Z}^d / \mathbb{Z}\mathbf{1})$ and actually

$$\{[L] : L \text{ is } \Lambda_M\text{-stable}\} \longleftrightarrow Q_M \cap (\mathbb{Z}^d / \mathbb{Z}\mathbf{1})$$



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Theorem. Let $\Gamma = \{L_{u^{(1)}}, \dots, L_{u^{(n)}}\}$ be any configuration of diagonal lattices in K^d . Then $\text{PZ}(\Gamma) = \Lambda_M$ where

$$M = M(u^{(1)}) \overline{\oplus} M(u^{(2)}) \overline{\oplus} \dots \overline{\oplus} M(u^{(n)}).$$



Definition.

- $\mathcal{P}_d = \{N \in \mathbb{R}_0^{d \times d} : N \underline{\odot} N = N\}$ is the **polytrope region**
- $\mathcal{P}_d(M) = \{N \in \mathcal{P}_d : N \leq M\}$ is the **truncated poly region**



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Remark.

- \mathcal{P}_d is a $(d^2 - d)$ -dimensional convex polyhedral cone, defined by “idempotence” inequalities $m_{ik} \leq m_{ij} + m_{jk}$
- The number of facets of \mathcal{P}_d is $d(d-1)(d-2)$
- For $d = 3, 4, 5$ we compute f -vectors and vertices of \mathcal{P}_d
- $\mathcal{P}_d(M)$ parametrizes all subpolytropes of Q_M
- $\mathcal{P}_d(M)$ parametrizes all \mathcal{O}_K -orders containing Λ_M

Indeed: $N \leq M \iff Q_N \subseteq Q_M \iff \Lambda_N \supseteq \Lambda_M$.

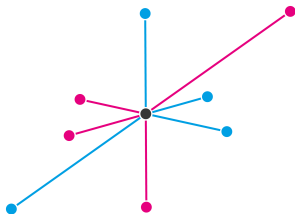
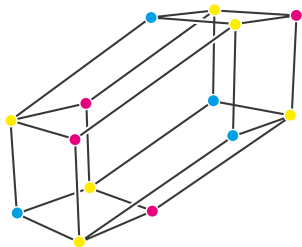


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Theorem. Let $M \in \mathcal{P}_d$ be in standard form. Then Q_M is both a min-plus and a max-plus simplex. The **min-plus** vertices u are the columns of M and represent L_u 's that are projective Λ_M -modules. The **max-plus** vertices v are the columns of $-M^t$, and they represent the injective Λ_M -modules L_v .



Here $d = 4$ and $M = J_4$ has all 1's outside the diagonal.



For M in standard form, the nonzero fractional ideals of Λ_M are

$$I_N = \{ X \in K^{d \times d} : \text{val}(X) \geq N \},$$

where $N \in \mathbb{Z}^{d \times d}$ with $N \underline{\odot} M = M \underline{\odot} N = N$. Write $N \in \mathcal{Q}_M$.



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Remark.

- $N \in \mathbb{Z}_0^{d \times d} \cap \mathcal{Q}_M \Rightarrow I_N = \Lambda_N$ is an order
- $N, N' \in \mathcal{Q}_M \Rightarrow I_N I_{N'} = I_{N \underline{\odot} N'}$
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Definition. The **ideal class group** \mathcal{G}_M of M is the maximal subgroup of the semigroup \mathcal{Q}_M .

Example. $\mathcal{G}_{J_2} \cong \mathbb{Z}/2\mathbb{Z}$, $\mathcal{G}_{J_3} \cong \mathbb{Z}/6\mathbb{Z}$, $\mathcal{G}_{J_4} \cong S_4$

Question. How does this sequence continue?



Definition. The Bruhat-Tits building $\mathcal{B}_d(K)$ is a simplicial complex where:

- the vertices are equivalence classes of lattices in K^d ,
- $([L_1], \dots, [L_s])$ is a simplex if $L_1 \supset L_2 \supset \dots \supset L_s \supset \pi L_1$
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Question. What's life like when you are not quarantined in one apartment?

thank you

And check out our Mathrepo page:

[https:](https://mathrepo.mis.mpg.de/OrdersPolytropes/index.html)

[//mathrepo.mis.mpg.de/OrdersPolytropes/index.html](https://mathrepo.mis.mpg.de/OrdersPolytropes/index.html)